On Recent Analytical Results for Solution of the Scattering Problem for Sharply Screened Coulomb Potentials

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- Solution of the problem for tail V^R potential
- Solution of the Coulomb problem in terms of core and tail solutions



List of Basic Notations

- Coulomb potential $V_{\mathcal{C}}(r)=Q/r,\ Q=\epsilon \frac{2\mu e^2 Z_1 Z_2}{\hbar^2}$ with $\epsilon=\pm 1$ corresponding to repulsion or attraction cases
- Schrödinger equation with Coulomb potential $[H_0 + V_{\mathcal{C}}(r) k^2] \psi_{\mathcal{C}}(r, k) = 0$
- Free Hamiltonian $H_0 = -\Delta_r$
- ullet Coulomb wave function $\psi_{\mathcal{C}}(r,k)=(2\pi)^{-3/2}e^{im{k}\cdotm{r}}e^{-\pi\eta/2}\Gamma(1+i\eta)_1F_1(-i\eta,1,i(kr-m{k}\cdotm{r}))$
- ullet Plane wave $\psi_0(oldsymbol{r},oldsymbol{k})=\exp\{ioldsymbol{r}\cdotoldsymbol{k}\}$
- ullet Sommerfeld parameter $\eta=Q/(2k)$
- ullet Coulomb phase shift $\sigma_\ell = \arg \Gamma(\ell+1+i\eta)$
- The regular Coulomb function $F_{\ell}(\eta,z)$, the irregular Coulomb function $G_{\ell}(\eta,z)$, the Coulomb spherical waves $u_{\ell}^{\pm}(\eta,z)=e^{i\sigma_{\ell}}[G_{\ell}(\eta,z)\pm iF_{\ell}(\eta,z)]$, Riccati-Bessel function $\hat{j}_{\ell}(z)=F_{\ell}(0,z)$, Riccati-Hankel function $\hat{h}_{\ell}^{+}(z)=u_{\ell}^{+}(0,z)$.



I. Splitting of the Coulomb potential into the core V_R and tail V^R potentials

The Coulomb potential

$$V_{\mathcal{C}}(r) = rac{Q}{r} \equiv V_{R}(r) + V^{R}(r)$$



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ight.$$





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The tail potential

$$V^R(r) = \left\{egin{array}{ll} 0 & r < R \ Q/r & r \geq R \end{array}
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II. Solution for the core potential V_R

The Schrödinger equation of the problem

$$\left[H_0+V_R(r)-k^2
ight]\psi_R(r,k)=0$$

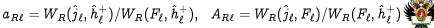
is solved by the partial wave expansion

$$\psi_R(r,oldsymbol{k}) = rac{1}{kr}\sum_{\ell\geq 0}(2\ell+1)i^\ell v_\ell(r,oldsymbol{k})P_\ell(\hat{oldsymbol{r}}\cdot\hat{oldsymbol{k}}).$$

The partial waves $v_{\ell}(r, k)$ are given by

$$v_\ell(r,k) = \left\{egin{array}{ll} a_{R\ell} F_\ell(\eta,kr) & r < R \ \hat{j}_\ell(kr) + A_{R\ell} \hat{h}^+_\ell(kr) & r \geq R \end{array}
ight.$$

where $a_{R\ell}$ and $A_{R\ell}$ are calculated through the Wronskians





Properties of the solution ψ_R

1) For r < R

$$\psi_R(oldsymbol{r},oldsymbol{k})=\int \mathsf{d}\hat{oldsymbol{k}}' a_R(\hat{oldsymbol{k}},\hat{oldsymbol{k}}')\psi_\mathcal{C}(oldsymbol{r},k\hat{oldsymbol{k}}')$$

with the kernel

$$a_R(\hat{m k},\hat{m k}') = \sum_{l>0} rac{2m \ell+1}{4\pi} a_{R\ell} e^{-i\sigma_\ell} P_\ell(\hat{m k}\cdot\hat{m k}').$$

Asymptotically, when $R o \infty$

$$a_{R\ell} \sim e^{i\sigma_\ell - i\eta \log 2kR}$$

Consequently,

$$a_R(\hat{m k},\hat{m k}')\sim e^{-i\eta\log 2kR}\delta(\hat{m k}-\hat{m k}')$$



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$$oldsymbol{\psi_R}(oldsymbol{r},oldsymbol{k}) \sim e^{-i\eta\log 2kR} oldsymbol{\psi_{\mathcal{C}}}(oldsymbol{r},oldsymbol{k})$$



Properties of the solution ψ_R

2) For r > R

$$\psi_R(oldsymbol{r},oldsymbol{k})=e^{ioldsymbol{r}\cdotoldsymbol{k}}+v_{sc}(oldsymbol{r},oldsymbol{k})$$

where

$$v_{sc}(r,m{k}) = rac{1}{kr} \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell A_{R\ell} \, \hat{h}^+_\ell(kr) P_\ell(\hat{m{k}}\cdot\hat{m{k}}').$$

Asymptotically, when $r o \infty$

$$v_{sc}(r,k) \sim A_R(u,k) e^{ikr}/r, \;\;\; u = \hat{m k} \cdot \hat{m k}'$$

with the amplitude

$$A_R(u,k) = rac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) A_{R\ell} P_\ell(u)$$

If $R o \infty$

$$egin{array}{ll} A_R(u,k) &\sim e^{-2i\eta\log 2kR}A_{\mathcal{C}}(u,k) \ &-rac{2}{k}\,e^{-i\eta\log 2kR}\sin(\eta\log 2kR)\,\delta(u-1) \end{array}$$



The Green function for V_R potential

The Green function $G^+(E) = (H_0 + V_R - E + i0)^{-1}$ is given by the partial wave representation

$$G_R^+(m{r},m{r'},k^2) = rac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) rac{G_{R\ell}(m{r},m{r'},k^2)}{m{r}m{r'}} P_\ell(m{\hat{r}}\cdotm{\hat{r'}})$$

For the most important configuration when $r, r' \leq R$ the partial wave components read

$$egin{array}{lll} G_{R\ell}(r,r',k^2) & = & rac{1}{k} F_{\ell}(\eta,kr_<) H^+_{\ell}(\eta,kr_>) + rac{\chi_{R\ell}(k)}{k} F_{\ell}(\eta,kr) F_{\ell}(\eta,kr'), \ & \chi_{R\ell}(k) & = & -W_R(\hat{h}^+_{\ell},H^+_{\ell})/W_R(\hat{h}^+_{\ell},F_{\ell}). \end{array}$$



The Green function for V_R potential

Summation of partial series in the region $r, r' \leq R$ leads to

$$G_R^+(m{r},m{r'},m{k}^2) = G_{\mathcal{C}}(m{r},m{r'},m{k}_+^2) + Q_R(m{r},m{r'},m{k}^2),$$

where $G_{\mathcal{C}}(r, r', k_{+}^{2})$ is the Coulomb Green function and Q_{R} is given by

$$egin{aligned} Q_R(r,r',k^2) &= rac{1}{2\mathrm{i}} \int_{-1}^1 \mathrm{d}\zeta \ Z_R(\xi,\zeta) [G_{\mathcal{C}}(r\hat{x},r'\hat{x}',k_+^2) - G_{\mathcal{C}}(r\hat{x},r'\hat{x}',k_-^2)]. \ k_\pm^2 &= k^2 \pm i0, \ \ \xi &= \hat{r} \cdot \hat{r}', \ \ \zeta &= \hat{x} \cdot \hat{x}' \ Z_R(\xi,\zeta) &= \sum_{\ell=0}^\infty (\ell+1/2) \chi_{R\ell}(k) P_\ell(\xi) P_\ell(\zeta) \ \chi_{R\ell}(k) &= -W_R(\hat{h}_\ell^+,H_\ell^+)/W_R(\hat{h}_\ell^+,F_\ell) \end{aligned}$$



The Green function for V_R potential

Asymptotically, when $R \to \infty$

$$\chi_{R\ell}(k)=\mathrm{i}\eta\exp(2\mathrm{i} heta_\ell)/(kR)+\mathcal{O}(1/R^2),$$

where $\theta_{\ell}=kR-\eta\log(2kR)-\pi\ell/2+\sigma_{\ell}$. For the $L_2(-1,1)$ norm of the kernel Z_R one gets

$$\|Z_R\|=\max_\ell |\chi_{R\ell}(k)|=\eta/(kR)+\mathcal{O}(R^{-2}).$$

Therefore

$$egin{split} Q_R(oldsymbol{r},oldsymbol{r}',oldsymbol{k}^2) &= \mathcal{O}(1/R) \ G_R(oldsymbol{r},oldsymbol{r}',oldsymbol{k}^2) &= G_\mathcal{C}(oldsymbol{r},oldsymbol{r}',oldsymbol{k}^2) + \mathcal{O}(1/R) \end{split}$$



T-matrix for V_R potential

Transition operator

$$T_R(z) = V_R - V_R G_R(z) V_R$$

Takes the form

$$T_R(z) = V_R - V_R G_{\mathcal{C}}(z) V_R - V_R Q_R(z) V_R.$$

Asymptotically, when $R \to \infty$

$$T_R(z) = V_R - V_R G_{\mathcal{C}}(z) V_R + \mathcal{O}(1/R)$$



III. Solution for the tail V^R potential

The Schrödinger equation of the problem

$$\left[H_0+V^R(r)-k^2
ight]\psi^R(r,k)=0$$

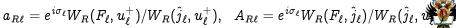
is solved by the partial wave expansion

$$\psi^R(r,m{k}) = rac{1}{kr}\sum_{\ell\geq 0}(2\ell+1)i^\ell w_\ell(r,m{k})P_\ell(\hat{m{r}}\cdot\hat{m{k}}).$$

The partial waves $w_{\ell}(r, k)$ are given by

$$w_{\ell}(r,k) = \left\{egin{array}{ll} a_{\ell}^R \hat{j}_{\ell}(kr) & r < R \ e^{i\sigma_{\ell}} F_{\ell}(\eta,kr) + A_{\ell}^R u_{\ell}^+(kr) & r \geq R \end{array}
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where a_{ℓ}^{R} and A_{ℓ}^{R} are calculated through the Wronskians





Properties of the solution ψ^R

1) For r < R

$$oldsymbol{\psi}^R(oldsymbol{r},oldsymbol{k}) = \int \mathrm{d} \hat{oldsymbol{k}}' a^R(\hat{oldsymbol{k}},\hat{oldsymbol{k}}') \psi_0(oldsymbol{r},k\hat{oldsymbol{k}}')$$

with the kernel

$$a^R(\hat{m k},\hat{m k}') = \sum_{l\geq 0} rac{2\ell+1}{4\pi} a_\ell^R P_\ell(\hat{m k}\cdot\hat{m k}').$$

Asymptotically, when $R o \infty$

$$a_\ell^R \sim e^{i\eta \log 2kR}$$

Consequently,

$$a^R(\hat{m k},\hat{m k}')\sim e^{i\eta\log 2kR}\delta(\hat{m k}-\hat{m k}')$$



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Asymptotically, when $R \to \infty$

$$a_\ell^R \sim e^{i\eta \log 2kR}$$

Consequently,

$$oldsymbol{\psi}^R(r,k) \sim e^{i\eta \log 2kR} oldsymbol{\psi}_0(r,k)$$



Properties of the solution ψ^R

2) For r > R

$$\psi^R(r,k) = \psi_{\mathcal{C}}(r,k) + w_{sc}(r,k)$$

where

$$w_{sc}(r,m{k}) = rac{1}{kr} \sum_{\ell>0} (2\ell+1) i^\ell A_\ell^R \, u_\ell^+(\eta,kr) P_\ell(\hat{m{k}}\cdot\hat{m{k}}').$$

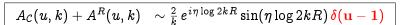
Asymptotically, when $r o \infty$

$$w_{sc}(r,k) \sim A^R(u,k) e^{i(kr-\eta \log 2kr)}/r, \;\; u = \hat{m k} \cdot \hat{m k}'$$

with the amplitude

$$A^R(u,k) = rac{1}{k} \sum_{\ell > 0} (2\ell+1) A_\ell^R P_\ell(u)$$

If $R \to \infty$





IV. Solution of the Coulomb problem in terms of core and tail solutions

The solution to the Coulomb Schrödinger equation

$$[H_0 + V_{\mathcal{C}}]\psi_{\mathcal{C}} = k^2\psi_{\mathcal{C}}$$

by the splitting procedure $V_{\mathcal{C}} = V^R + V_R$ can be represented as

$$oldsymbol{\psi}_{\mathcal{C}} = oldsymbol{\psi}^{\mathbf{R}} - [H_0 + V_{\mathcal{C}} - k^2 + i0]^{-1} V_{R} oldsymbol{\psi}^{\mathbf{R}}$$

or equivalently in the form of the integral equation of the Lippmann-Schwinger type

$$\psi_{\mathcal{C}} = {m \psi}^{m R} - [H_0 + V^R - k^2 + i0]^{-1} V_R \psi_{\mathcal{C}}$$



IV. Solution of the Coulomb problem in terms of core and tail solutions

Asymptotically, when $r \to \infty$

$$\psi_{\mathcal{C}}(r,k) \sim \psi^{R}(r,k) + F_{R}e^{i(k\,r-\eta\log2k\,r)}/r$$

where the amplitude is defined as

$$F_R = -2\pi^2 \langle oldsymbol{\psi}^{R(-)}(oldsymbol{k}')|\,V_R\,|oldsymbol{\psi}_{\mathcal{C}}(oldsymbol{k})
angle.$$

Here $\psi^{R(-)}(oldsymbol{r},oldsymbol{k}')=(\psi^R(oldsymbol{r},-oldsymbol{k}'))^*.$

The further representation for the amplitude F_R has the form

$$egin{aligned} F_R &= -2\pi^2 [\langle oldsymbol{\psi}^{R(-)}(oldsymbol{k}') | T_R(oldsymbol{k}^2 + i0) | oldsymbol{\psi}^R(oldsymbol{k})
angle \ &+ \langle oldsymbol{\psi}^{R(-)}(oldsymbol{k}') | Q_R(oldsymbol{k}^2 + i0) | oldsymbol{\psi}^R(oldsymbol{k})
angle] \end{aligned}$$



IV. Solution of the Coulomb problem in terms of core and tail solutions

Asymptotically, when $R \to \infty$

$$F_R = -2\pi^2 e^{2i\eta \log 2kR} \langle \psi_0(oldsymbol{k}')|T_R(oldsymbol{k}^2+i0)|\psi_0(oldsymbol{k})
angle + \mathcal{O}(1/R)$$

and then

the total Coulomb amplitude receives the representation

$$egin{array}{lll} A_{\mathcal{C}} &=& rac{2}{k} e^{i\eta\log 2kR} \sin(\eta\log 2kR) rac{\delta(\mathbf{u}-\mathbf{1})}{\delta(\mathbf{u}-\mathbf{1})} \ &-& 2\pi^2 e^{2i\eta\log 2kR} \langle \psi_0(m{k}') | T_R(k^2+i0) | \psi_0(m{k})
angle + \mathcal{O}(1/R) \end{array}$$



RÉSUMÉ

- The splitting approach allows to treat the long-range scattering problem on the basis of the short-range formalism
- The use of the splitting approach for systems of $N \geq 3$ particles seems to be very promising



Publications

- S.L. Yakovlev, M.V. Volkov, E. Yarevsky and N. Elander, The Impact of Sharp Screening on the Coulomb Scattering Problem in Three Dimensions J. Phys. A: Math. Theor. 43 (2010) 245302.
- M. V. Volkov, S. L. Yakovlev, E. A. Yarevsky, and N. Elander, Potential splitting approach to multichannel Coulomb scattering: The driven Schrödinger equation formulation Phys. Rev. A 83 (2011) 032722
- S.L. Yakovlev, V.A. Gradusov, Zero range potential for particles interacting via Coulomb potential: application to electron-positron annihilation J. Phys. A: Math. Theor. 46 (2013) 035307



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THANK YOU FOR YOUR ATTENTION





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