On the importance of the tail of proton charge density

or: how to get the *rms*-radius from (e,e) data?

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Proton *rms*-radius

important quantity
needed to interpret hyper-precise atomic transition energies
traditionally determined via electron scattering, $q = 0$ slope of $G(q^2)$
analysis of world data yields R=0.886±0.008 fm

Recent result from Lamb-shift in muonic Hydrogen

very precise radius: R=0.8418±0.0007 fm
disagrees with (e,e) by many σ

Reasons for discrepancy?

many ideas discussed in literature
  too many to detail here
no culprit identified

Investigated here:

scrutinize determination of *rms*-radius from (e,e)-data

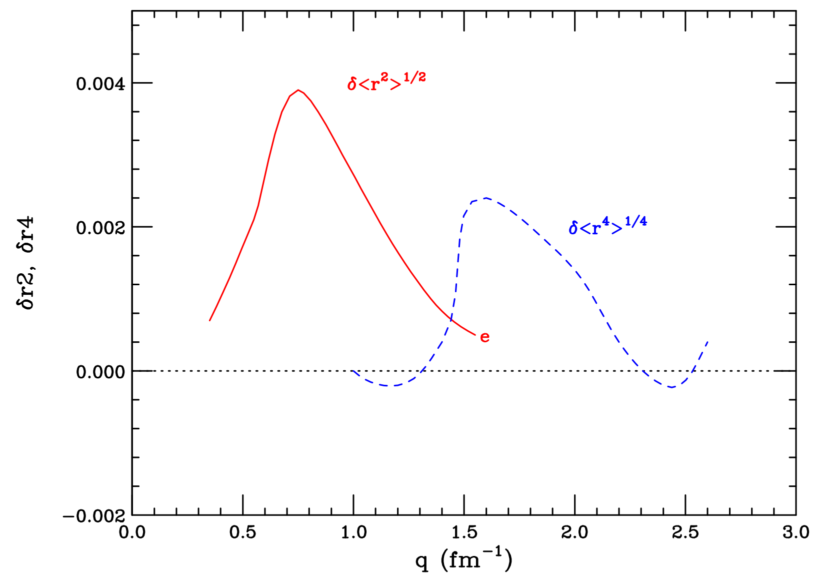
How to determine the *rms*-radius?

priori this looks simple:

fit data with parameterization for $G_e(q)$, $G_m(q)$
$q = 0$ slope of $G_e(q) \rightarrow$ *rms*-radius R

An unavoidable problem:

cannot measure down to $q = 0$
even if could, finite size effect too small: $G(q) = 1 - q^2 R^2/6 + ...$
at very low q measure only the "1"
given exp. uncertainties δG



q-region sensitive to *rms*-radii
$0.5 < q < 1.3 fm^{-1}$
$0.01 < Q^2 < 0.06 GeV^2/c^2$

extrapolation to $q = 0$ introduces model dependence
particularly bad for proton

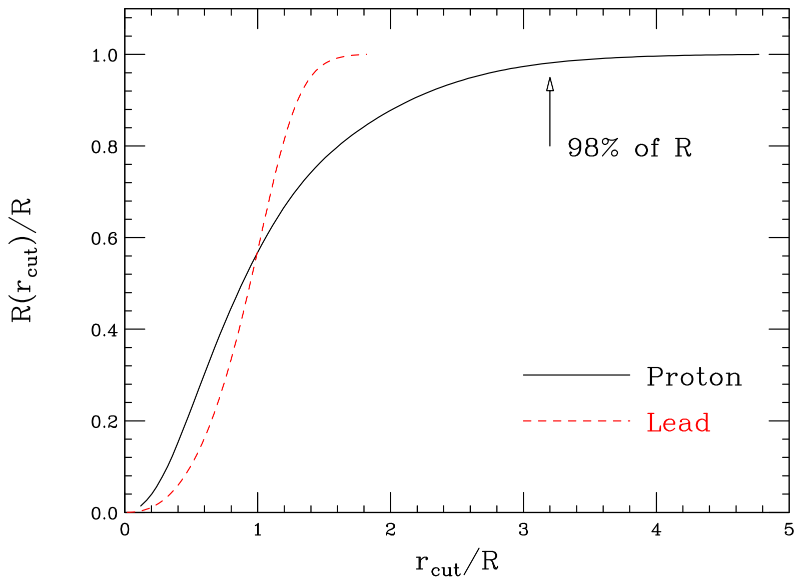
Proton = particularly difficult case

form factor $\sim$ dipole $1/(1+q^2c^2)^2$
$\rightarrow$ density $\sim$ exponential $\sim e^{-r/c}$

exponential density has very long tail!

Study $[\int_0^{r_{cut}} \rho(r)\ r^4 dr / \int_0^{\infty} \rho(r)\ r^4 dr]^{1/2}$ as function of cutoff r_{cut}

both normalized to rms-radius R



to get 98% of R must integrate out to $r \sim 3 \cdot R \sim 3fm$

Consequence: R sensitive to $\rho(r)$ at very large r where $\rho(r)$ poorly determined
affects $G(q)$ at very low q, below q_{min}

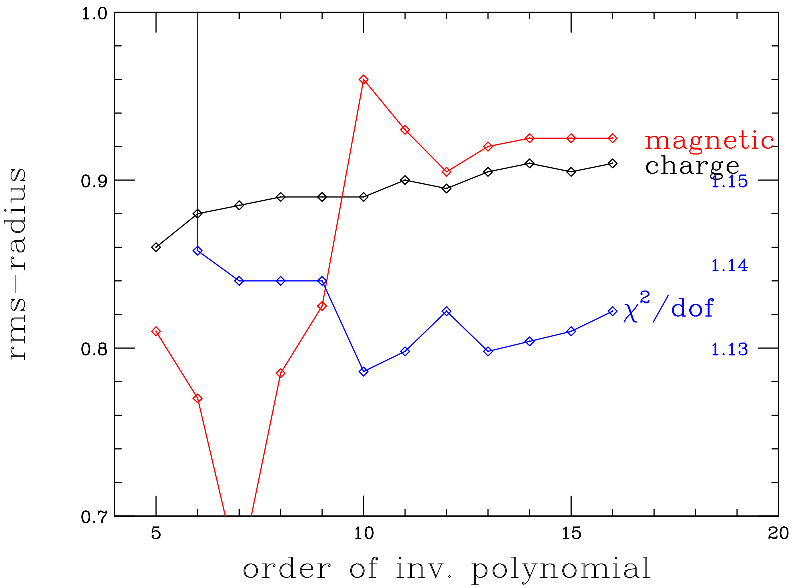
But there are worse pitfalls!

discuss starting from two recent results:

- Inverse Polynomial fit of Bernauer *et al.*
- Continued Fraction fit of Lorenz *et al.*

Inverse Polynomial Bernauer

$$G(q^2) = 1/(1 + a_1 q^2 + a_2 q^4 + ...)$$



Curious behavior:

between order N=7 and N=10 R^M jumps from $0.68 fm$ to $0.96 fm$

χ^2 best for N=10

would nominally be *the* best fit!

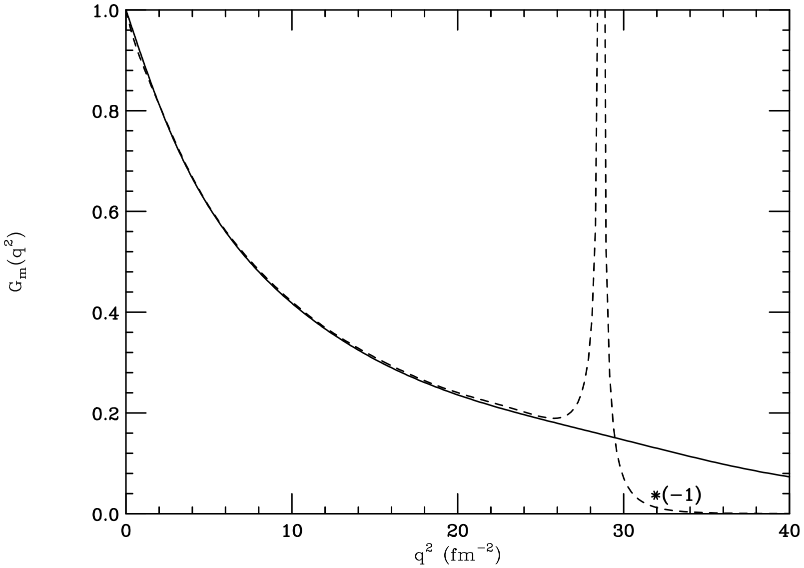
Bernauer *et al.* chose order N=7 (χ^2 ±stabilized)

Question remains:

what is responsible for jump?

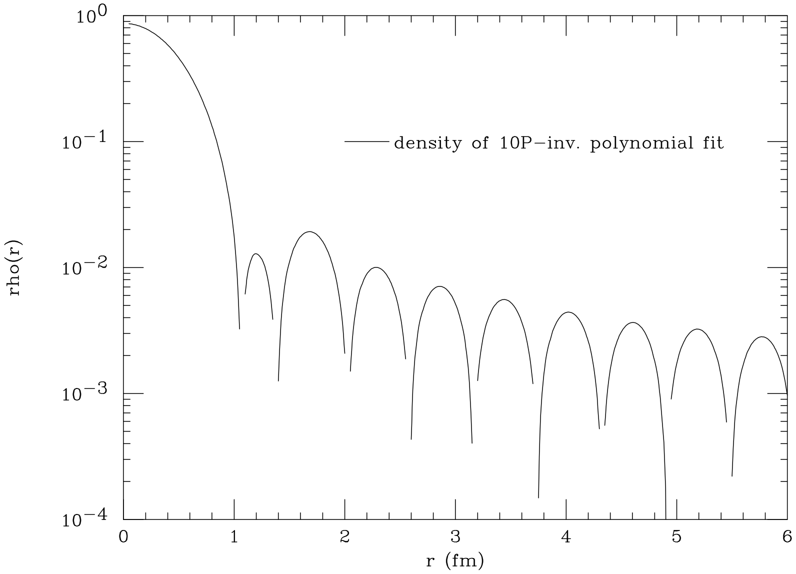
how can the q^{20}-term affect the rms-radius?

Understanding

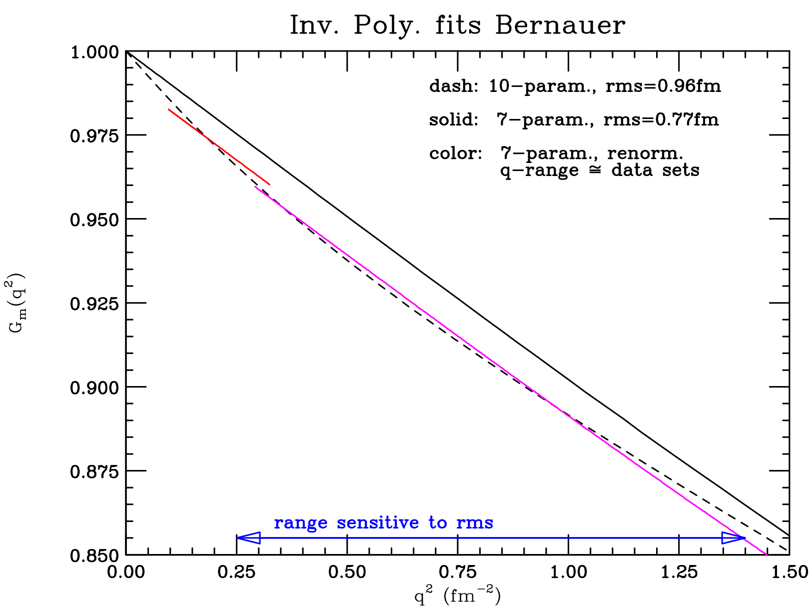


G_M for N=10 has pole at $q > q_{max}$

In $\rho(r)_m$ this leads to oscillations extending to *very* large r



This affects $G_m(q^2)$ at *very low* q^2, below q^2_{min} of data



Structure at $q < q_{min}$ gives better χ^2 than N=7 (note: data are floating)

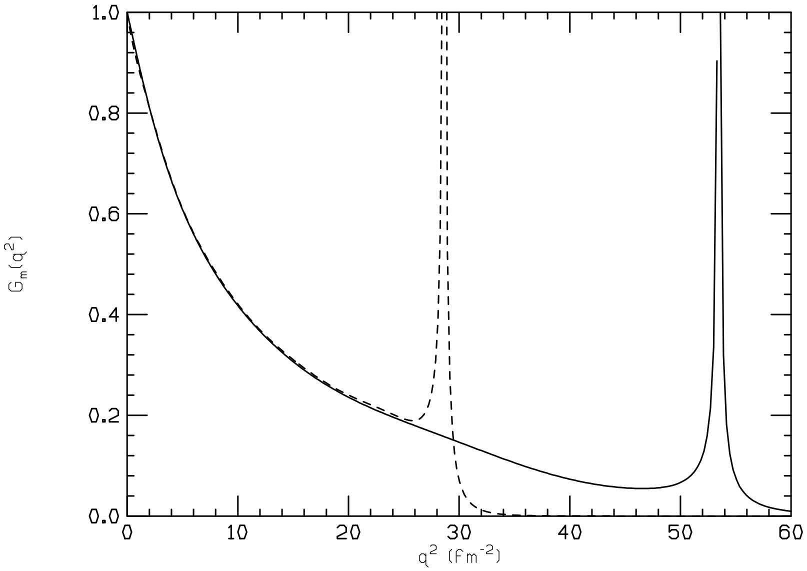
confirms old insight that absolute σ's *much* more valuable than floating ones

Conclusion: N=10 fit is pathological.

but is N=7 better?

A priori: yes, since more 'reasonable'

however: N=7 has pole too!



but pole is at larger q, happens to have much smaller effect

Cannot believe either radius!

Continued Fraction fits by Lorenz *et al.*

$$G(q) = \cfrac{1}{1 + \cfrac{q^2 b_1}{1 + \cfrac{q^2 b_2}{1 + \cdots}}}$$

many fits of Bernauer data with variable q_{max}

for *e.g.* 5 terms and $q_{max} = 3.5 fm^{-1}$ find charge-rms-radius $0.84 fm$

disagrees with "accepted" result of $0.88 fm$

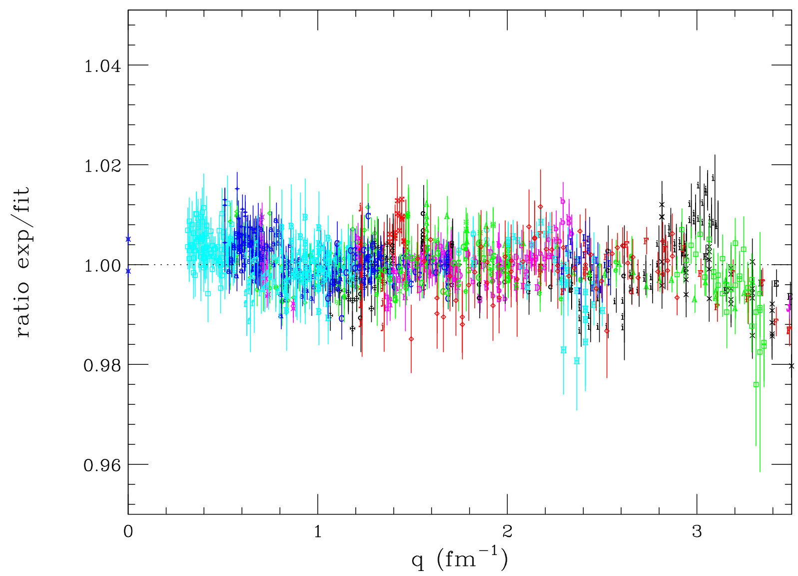
One reason

$\chi^2 \sim 1.4/dof$ not very good

$\rightarrow$ systematic deviations at low q

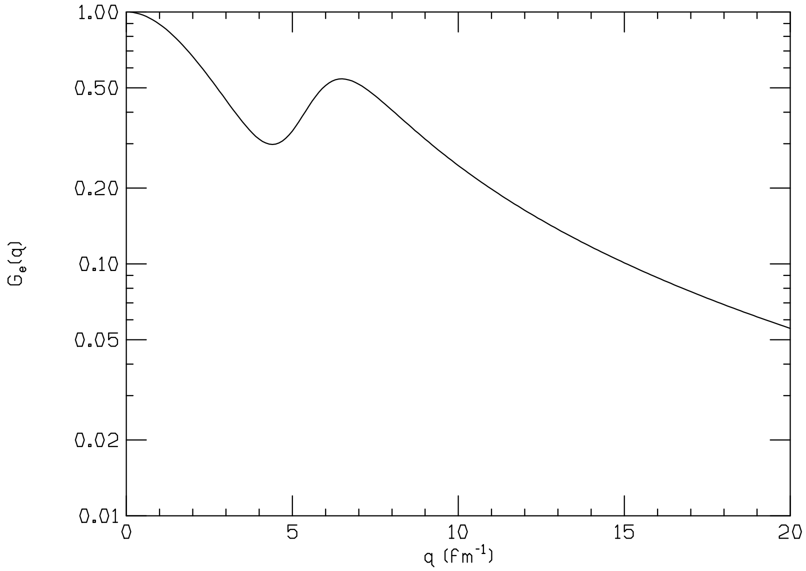
Spline fit gives 1.06/dof

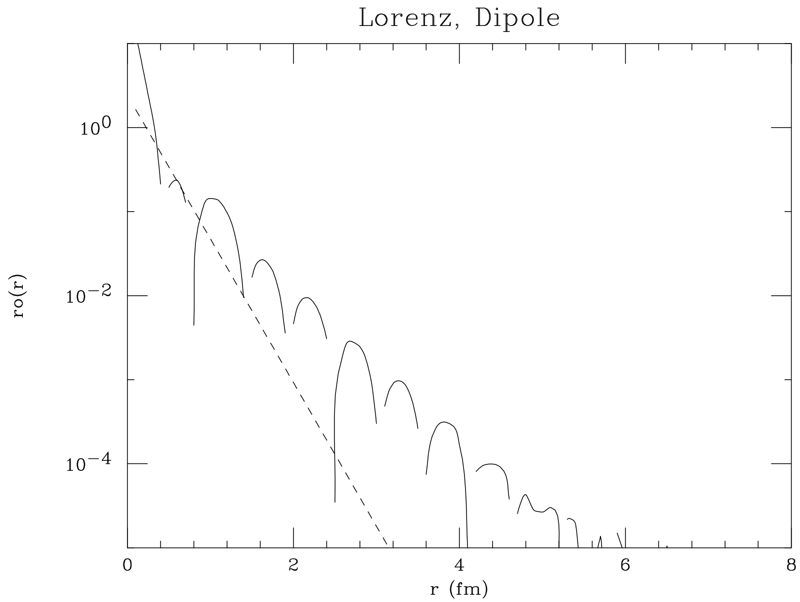
from such a fit cannot draw conclusions



Main problem of Lorenz *et al.*

Unphysical behavior of G at $q > q_{max} = 3.5 fm^{-1}$ (parameters courtesy H.-W. Hamm





large $G(q)$ at large q

falls *very* slowly

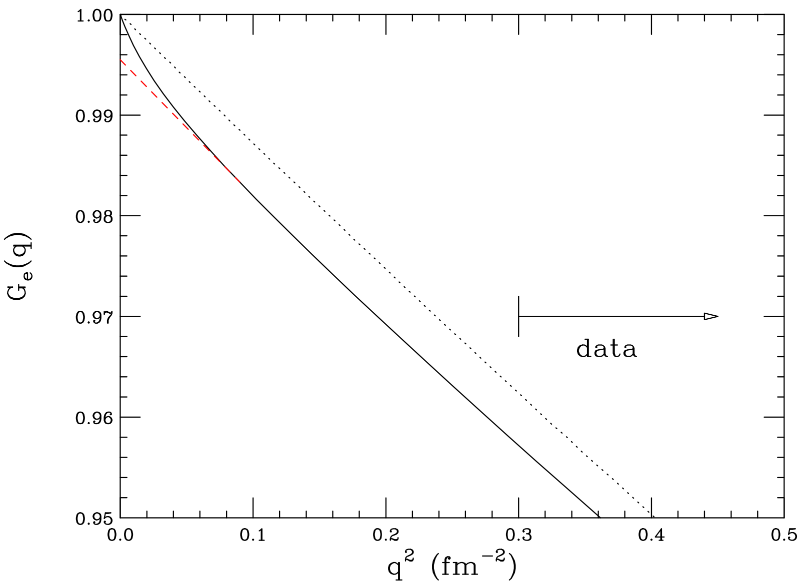
$\rightarrow$ structure of $\rho(r)$ at very large r

large contribution to rms-radius

affecting $G(q < q_{min})$

Extreme demonstration case

own 3-parameter Pade-fit of Bernauer data, $q < 2fm^{-1}$
excellent $\chi^2 \sim 1.06$/dof
no pole



——— Pade-fit
.... Fit with 'normal' rms-radius

rms-radius $= 1.49 fm$!!

visible by nude eye in $G(q)$ at *very* low q
problem enhanced by floating data

Again: problem due to uncontrolled behavior of $G(q > q_{max})$

leads to structure of $G(q < q_{min})$ affecting $q = 0$ slope

For understanding: compare approach for $A > 2 \quad \Longleftrightarrow \quad A \leq 2$

- for A>2 parameterize $\rho(r)$, fit to data, get rms-radius from integral over r

- for A$\leq$ 2 parameterize $G(q)$, fit to data, get rms-radius from $q = 0$ slope

Not equivalent !!

- $\rho(r)$ automatically confined to $r < r_{max}$ by parameterization
  Fermi density, Gauss density, Fourier-Bessel, SOG, ...
  good physics: $\rho(r)$ must fall like $W(\kappa r)^2/r^2$
  κ given by removal energy of lightest, least bound charged constituent

- constraint is missing when parameterizing $G(q)$
  $G(q > q_{max})$ *can* imply large ρ at large r
  allows for unphysical structure of $G(q)$ below q_{min}
  can falsify rms-radius

This is a generic problem. MUST be avoided.

...... and unfortunately concerns most current fits

Least affected: fits including data up to maximal q of data
in this case data fix large-r behavior pretty well
$G(q > q_{max})$ constrained by small $G(q \sim q_{max})$ if $G \sim q^{-4}$ ($\rightarrow$ regular $\rho(0)$)

"Solutions"

1. Parameterize $G(q)$, always compute $\rho(r)$, check large-r behavior
   difficult as above examples show, not possible for parameterizations without FT

2. Parameterize $\rho(r)$ with sensible large-r fall-off
   FT$\rightarrow$ $G(q)$, fit parameters to σ's
   complicates life, but only a bit. Tricky point: definition of "sensible"

Solution

- parameterize $\rho(r)$ in basis with analytic FT
  SOG, Hermite, Laguerre, ...

- constrain $\rho(r \gg)$ using physical model, for r where $\rho(r) < 0.01 \cdot \rho(0)$
  fall-off of ρ given by least-bound Fock component of proton = n+π^+
  (+complications if desired, see I.S., Prog.Part.Nucl.Phys. 67 (2012)473)
  $\rightarrow$ adds physics explicitly, safest choice!

- fit data up to maximal q, so data constrain tail of ρ as well
  straightforward with above bases

To conclude

Bad news

parameterized $G(q)$'s *may* have problems

very difficult to identify if this is the case or not

particularly if $G(q)$ has no FT (such as popular sum of powers of q^2)

$q = 0$ slope could be right or wrong

$q = 0$ slope could be sensible or not

can be believed only if $\rho(r)$ at large r has been studied! ..... which is never don

Good news

r-space fit with large-r constraint gives stable radii, free of diseases discussed

For data=world, =world+Bernauer(+.4%), =world with fixed or floating norm, find

$R_{ch} = 0.886 \pm .008 fm$, $R_m = 0.858 \pm .024 fm$

see I.S., Prog. Part. Nucl. Phys. 67 (2012) 473

.......unfortunately it does not help with μH discrepancy