Solutions of the Bethe-Salpeter Equation in Minkowski space: a comparative study

G.S.

INFN - Rome

In collaboration with

Tobias Frederico (ITA - S. José dos Campos)
Michele Viviani (INFN - Pisa)

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Outline

1. Motivations and generalities: BS Amplitude and BS Equation for a two-scalar bound system $\rightarrow \mathcal{L} = g\phi^2\chi$
2. Nakanishi perturbation-theory integral representation (PTIR) and the BS Amplitude
3. The exact projection of the BSE onto the null plane and the PTIR of BSA
4. Eigenvalues and LF distributions in ladder approximations
5. Conclusions & Perspectives

Motivations

- To achieve a fully covariant description for a few-body system, in Minkowsky space
- To take into account properly the dynamics, within a field-theoretical framework
- To make feasable numerical calculations

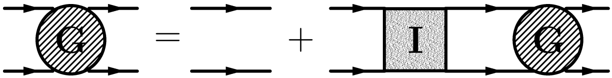
Well-known non perturbative approches: lattice calculations in Euclidean space

The BSE in a nutshell

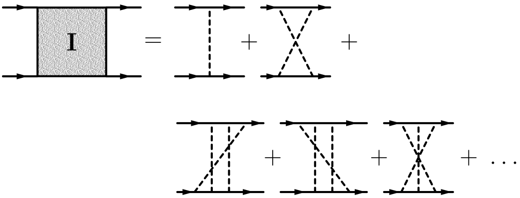
The 4-point Green's Function,

$$G(x_1, x_2; y_1, y_2) = < 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 > \quad ,$$

fulfills an integral equation $G = G_0 + G_0 \, \mathrm{I} \, G$



I $\equiv$ kernel given by the infinite sum of irreducible Feynmann graphs



Iterations produce all the expected contributions

Insert a complete Fock basis in

$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 >$$

then in the Fourier space, the bound state contribution (assuming only one non degenerate bound state for the sake of simplicity) appears as a pole, i.e.

$$G_B(k, q; p_B) \simeq \frac{i}{(2\pi)^{-4}} \frac{\phi(k; p_B)\ \bar{\phi}(k; p_B)}{2\omega_B(p_0 - \omega_B + i\epsilon)}$$

where $\omega_B = \sqrt{M_B^2 + |\mathbf{p}|^2}$ and $\phi(k; p_B)$ is obtained from the Fourier transform of the Bethe-Salpeter Amplitude for a bound state given by

$$\langle 0| T\{\phi_1(x_1)\phi_2(x_2)\}|p_B\, \beta\rangle$$

For $p_0 \to \omega_B$ the 4-point Green's function is given by

$$G \simeq G_B + \; regular \;\; terms$$

and one deduces from $G = G_0 \; + \; G_0 \; \mathrm{I} \; G$, the integral equation determining the BS Amplitude for a bound state, i.e.

$$\phi(k; p_B, \beta) = G_0(k; p_B, \beta) \; \int d^4q' \; \mathrm{I}(k, q'; p_B) \; \phi(q'; p_B, \beta)$$

with

$$G_0 = \frac{i}{(\frac{p_B}{2} + k)^2 - m^2 + i\epsilon} \;\; \frac{i}{(\frac{p_B}{2} - k)^2 - m^2 + i\epsilon}$$

Notably, the same irreducible kernel, $\mathrm{I}(k, q'; p_B)$, is acting.

Feynman parametrization

In the sixties, Nakanishi (PR **130**, 1230 (1963)) proposed an integral representation for N-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.
For N external legs, a generic contribution to the transition amplitude is given by

$$f_{\mathcal{G}}(p_1, p_2, ..., p_N) \propto \prod_{r=1}^{k} \int d^4 q_r \frac{1}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2) \ \ldots \ (\ell_n^2 - m_n^2)}$$

where one has n propagators and k loops ($\equiv$ n. of integration variables).
The label $\mathcal{G} \to (n, k)$

Following the standard (textbook) elaboration, one can write

$$f_{\mathcal{G}}(s) \propto \prod_{i=1}^{n} \int_0^1 d\alpha_i \; \frac{\delta(1 - \sum_{j=1}^{n} \alpha_j)}{U^2(\alpha) \; [F(n, N, \alpha, s) + i\epsilon]^{n-2k}}$$

where

$$F(n, N, \alpha, s) = -\sum_{j=1}^{n} \alpha_j \; m_j^2 + \sum_{h} \; \eta_h \; s_h$$

with the dependence upon the external momenta, $p_1, p_2 \dots p_N$ traded off in favour of all the independent scalar products $s \equiv \{s_1, s_2, \dots s_h, \dots\}$, one can construct.

Nakanishi PTIR - I

Nakanishi proposal for a compact and elegant espression of the full N-leg amplitude $f_N(s) = \sum_{\mathcal{G}} \; f_{\mathcal{G}}(s)$

Introducing the identity

$$1 \doteq \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\beta}\right) \int_0^\infty d\gamma \; \delta\left(\gamma - \sum_l \frac{\alpha_l m_l^2}{\beta}\right)$$

with $\beta = \sum \eta_i$ and integrating by parts $n - 2k - 1$ times

$$f_{\mathcal{G}}(s) \propto \; \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h) \; \tilde{\phi}_{\mathcal{G}}(z,\gamma)}{(\gamma - \sum_h z_h s_h)}$$

where $\tilde{\phi}_{\mathcal{G}}(z,\gamma)$ is a proper function

The depedence upon the details of the diagram, (n,k), moves from the denominator to the numerator!! The SAME formal expression for the denominator of ANY diagram $\mathcal{G}$ appears

Nakanishi PTIR - II

The full N-leg transition amplitude can be formally written as

$$f_N(s) \sum_{\mathcal{G}} f_{\mathcal{G}}(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h) \; \phi_N(z, \gamma)}{(\gamma - \sum_h z_h s_h)}$$

where

$$\phi_N(z, \gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(z, \gamma)$$

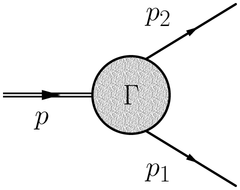
Within the BS framework, such an elegant expression can be exploited for obtaining

- the 3-leg transition amplitude (vertex function $\rightarrow$ bound-state BS amplitude) (Kusaka et al, PRD **56** (1997), Carbonell-Karmanov EPJA **27** (2006))
- the 4-leg one (off-shell or half-off-shell T-matrix $\rightarrow$ scattering-state BS amplitude) (FSV, PRD **85** (2012))

The PTIR of the vertex function

$$f_3(s) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk \cdot p - i\epsilon}$$

with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$



How can the Nakanishi weight function, ϕ_3, be determined for an actual, dynamical model?

Can the Nakanishi expression, elaborated in perturbation theory , be used in a non perturbative realm, as the BS framework is (one has to face with an integral equation)?

Integrating the BSE on the LF variable k^-

Let us take the Nakanishi vertex function as an Ansatz for the BS amplitude and then, integrate it on the Light-Front variable $k^- = k^0 + k_z$.

One gets the valence component of the state of the interacting system (after expanding on the Fockbasis)

$$\psi_{n=2}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}}\, \xi\, (1-\xi) \int \frac{dk^-}{2\pi} \Phi_b(k,p) =$$

$$= \frac{1}{\sqrt{2}} \xi\, (1-\xi) \int_0^\infty d\gamma' \, \frac{g_b(\gamma', 1-2\xi; \kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi-1)^2 \frac{M^2}{4} - i\epsilon]^2}$$

Applying the LF projection to both sided of the BSE, one ends up with

$$\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1-z^2)\kappa^2 - i\epsilon]^2} =$$

$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' \; V_b^{LF}(\gamma, z; \gamma', z') g_b(\gamma', z'; \kappa^2).$$

with $V_b^{LF}(\gamma, z; \gamma', z')$ determined by the irr. kernel $\mathrm{I}(k, k', p)$!

First obtained by Carbonell and Karmanov within the so-called explicitly-covariant LF framework (EPJA 27 (2006)).

Applying the uniqueness of the Nakanishi weight function

Nakanishi enriched his theoretical invetigation by demonstrating a theorem on the uniqueness of the weight function for a given N-leg amplitude.

If such a theorem is valid also in the non perturbative context of the BSE a simpler integral equation for the weight function can be written

$$g_b(\gamma, z; \kappa^2) = \int_0^\infty d\gamma' \int_{-1}^{1} dz' \, \mathcal{V}_b(\gamma, z; \gamma', z'; \kappa^2) g_b(\gamma', z'; \kappa^2)$$

where $\mathcal{V}_b(\gamma, z; \gamma', z'; \kappa^2)$ is a new kernel, properly related to $V_b^{LF}(\gamma, z; \gamma', z')$!

First in a canonical approach (Kusaka et al PRD **56**, (1997)), recently in a LF approach (FSV PRD **85**,(2012))

Numerical results for the eigenvalues

We have carried out a comprehensive investigation, in ladder approximation, of the simple scalar model, $\mathcal{L} = g\phi^2\chi$, i) varying both binding energies

$$0 < \frac{B}{m} \leq 2$$

and the mass of the exchanged scalar, and using ii) the two eigen-equations: the one involving directly the valence wave function and the one based on the uniqueness theorem.

One fixes the binding energy and the mass of the exchanged scalar, and looks for the eigenvalue (the coupling constant) and the eigenfunction (the Nakanishi weight function).

Comparison with the results from i) Carbonell-Karmanov (EPJA **27**, 1 (2006)) (valence w.f. based and covariant LF) and ii) Kusaka et al, (PRD **56**, 5071 (1997)) (uniqueness based and canonical approach). .

$\mu/m = 0.15$

B/m	α LF-V (CK)	α LF-V (FSV)	α LF-U (FSV)
0.01	0.5716	0.5716	0.5716
0.10	1.437	1.437	1.437
0.20	2.100	2.099	2.099
0.50	3.611	3.610	3.611
1.00	5.315	5.313	5.314

$\mu/m = 0.50$

B/m	α LF-V (CK)	α LF-V (FSV)	α LF-U (FSV)
0.01	1.440	1.440	1.440
0.10	2.498	2.498	2.498
0.20	3.251	3.251	3.251
0.50	4.901	4.901	4.901
1.00	6.712	6.711	6.711

Values of $\alpha = g^2/(16\pi m^2)$, obtained by solving the valence-based eigenequation (LF-V) and the uniqueness-based one (LF-U). Gegenbauer $\times$ Laguerre expansion of the Nakanishi wf

LF-V (CK): from Carbonell -Karmanov, EPJA **27**, 1 (2006) (spline expansion of the Nakanishi wf).

$\mu/m = 0.50$

B/m	α C-U	α LF-U	α LF-V
0.002	1.211	1.216	1.216
0.02	1.624	1.623	1.623
0.20	3.252	3.251	3.251
0.40	4.416	4.415	4.416
0.80	6.096	6.094	6.094
1.20	7.206	7.204	7.204
1.60	7.850	7.849	7.849
2.00	8.062	8.061	8.061

Values of $\alpha = g^2/(16\pi m^2)$, obtained by solving the valence-based eigenequation (LF-V) and the uniqueness-based one (LF-U). Gegenbauer $\times$ Laguerre expansion of the Nakanishi wf

C-U: from Kusaka,Simpson and Williams, PRD **56**, 5071 (1997), where uniqueness and canonical (not LF !) variables have been used and iteration method for solving the eigenequation.

Valence Probabilities and LF Distributions

Once the Nakanishi weight functions is evaluated, one can straightforwardly obtain the BS amplitude and normalize it.

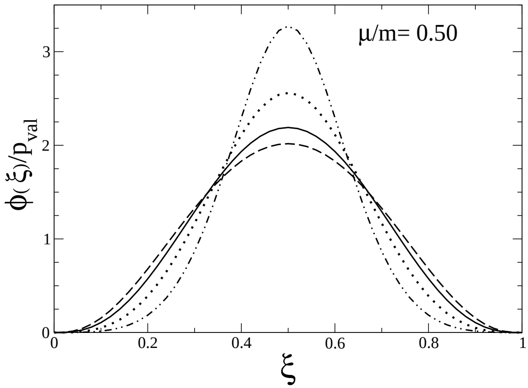
Then, the probability of the the valence wave function, $\psi_{n=2}(\xi, k_\perp)$, results properly determined and one can calculate the LF distributions, as well

$\mu/m = 0.50$

B/m	α	P_{val}
0.001	1.167	0.98
0.01	1.440	0.96
0.10	2.498	0.87
0.20	3.251	0.83
0.50	4.900	0.77
1.00	6.711	0.74
2.00	8.061	0.72

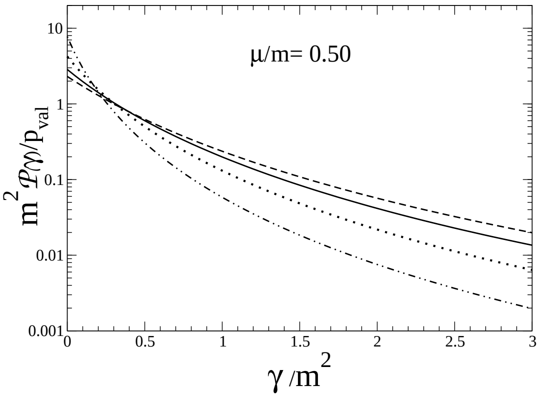
$P_{val} \rightarrow 1$ for $B \rightarrow 0$!

NO sizable difference between LF-V and LF-U results !!



The longitudinal LF-distribution, $\phi(\xi) = \int dk_\perp^2 \ |\psi_{n=2}(\xi, k_\perp)|^2$, vs the longitudinal-momentum fraction $\xi = k^+/M$. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. N.B. $\int_0^1 d\xi \ \phi(\xi) = P_{val}$

NO sizable difference between LF-V and LF-U results !!



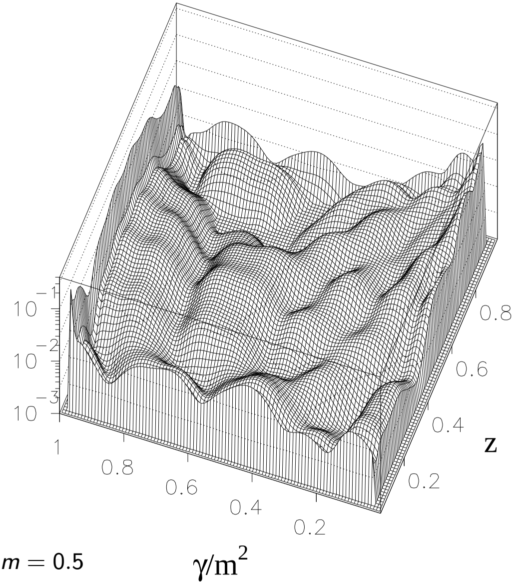
The transverse LF-distribution $\mathcal{P}(\gamma) = \int d\xi \ |\psi_{n=2}(\xi, k_\perp)|^2$ vs the adimensional variable γ/m^2 ($\gamma = k_\perp^2$). Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. N.B. $\int_0^\infty d\gamma \ \mathcal{P}(\gamma) = P_{val}$.

Conclusions & Perspectives

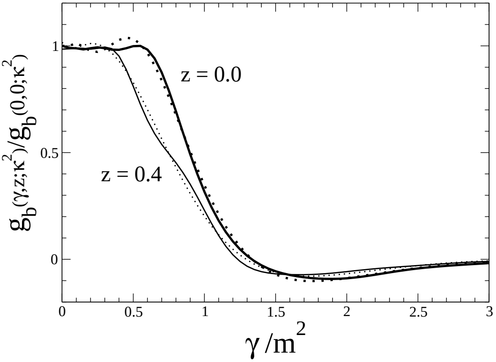
- The cross-fertilization between the Light-Front framework and the Nakanishi PTIR paves the path toward a new class of non perturbative calculations, within a rigorous field-theoretical framework (the Bethe-Salpeter Equation in Minkowski space).
- The LF framework has well-known advantages in performing analytical integrations, that within the canonical approach appear highly non trivial.
- Our numerical investigations, performed in ladder approximation at the present stage, confirm the robusteness of the Nakanishi Ansatz for the BS amplitude and the Uniqueness Theorem. Morever, we extended the numerical analysis of an actual dynamical model to the valence probability and the LF distributions, of great relevance for the Hadronic Physics.
- Calculations are in progress for i) the scattering length and ii) the crossed-box contribution with uniqueness th.

Back-up slides

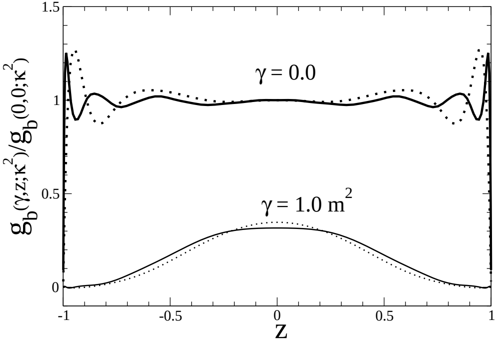
$$\Delta g(z,\gamma;\kappa^2) = \left| g_U(z,\gamma;\kappa^2) - g_V(z,\gamma;\kappa^2) \right|$$



$B/m = 1$ and $\mu/m = 0.5$



$B/m = 1$ and $\mu/m = 0.5$



$B/m = 1$ and $\mu/m = 0.5$