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Transition e.m. form factor in the Minkowski space Bethe-Salpeter approach

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• BS approach

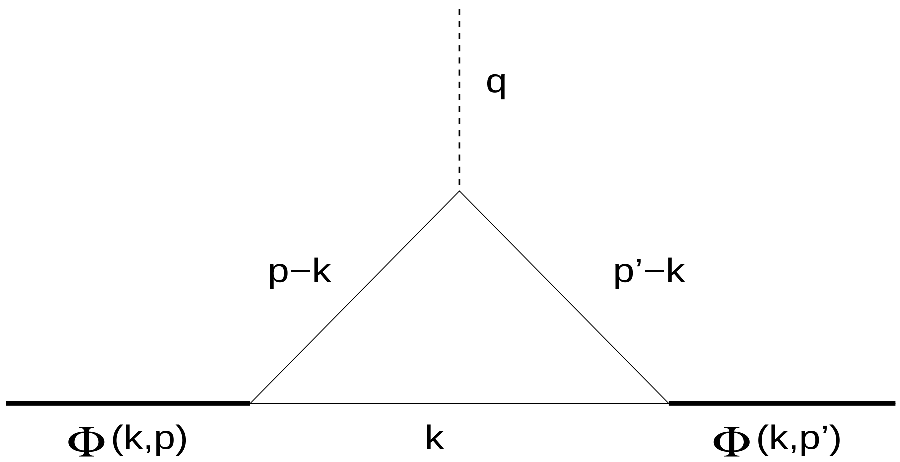
E.E. Salpeter and H. Bethe, Phys. Rev. **84**, 1232 (1951)

BS approach is a powerful tool
in relativistic few-body physics
and in field theory

To analyze a relativistic system (like a non-relativistic one) we need:

- Bound state BS amplitudes in Minkowski space –done
- Elastic E.M. form factors – done
- Scattering state BS amplitudes in Minkowski space –done
- Transition E.M. form factors (bound state $\to$ scattering state) –subject of this talk
- Relativistic Faddeev-Yakubovski amplitudes –not yet

• Why Minkowsky space?



E.m. vertex in terms of the BS amplitude.

$$(p+p')^\mu F_M(Q^2) = -i\int \frac{d^3k\,dk_0}{(2\pi)^4}(p+p'-2k)^\mu\,(m^2-k^2)\Phi_M\left(\frac{p}{2}-k,p\right)\Phi_M\left(\frac{p'}{2}-k,p'\right)$$

Wick rotation cannot be done in the form factor integral because of singularities of Φ_M vs. k_0.

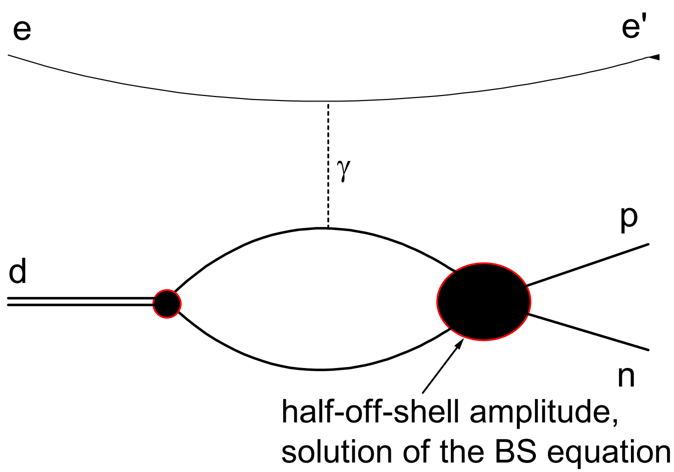
• Transition form factor

We need the

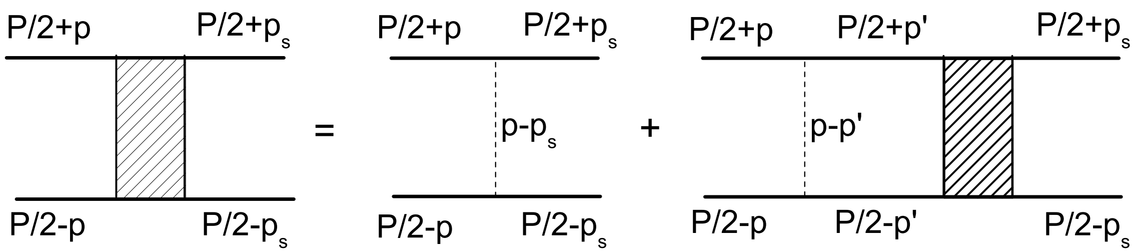
off-shell BS amplitudes in Minkowski space

(both for bound and scattering states)

to calculate the transition form factor $ed \to enp$.



• BS equation



$$F(p,p_s,P) \quad = \quad V^{inh}(p,p_s,P) - i\int \frac{d^4p'}{(2\pi)^4}$$

$$\times \quad \frac{V(p,p',P)F(p',p_s,P)}{\left[\left(\frac{1}{2}P+p'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{1}{2}P-p'\right)^2 - m^2 + i\epsilon\right]}$$

Scattering state mass: $M \equiv \sqrt{s} = 2\sqrt{m^2+p_s^2} > 2m$

Equation determines the off-shell amplitude in Minkowski space.

In c.m.-frame $\vec{P}=0$: $F_l = F_l(p_0, p; p_s)$ depends on two variables p_0, p (for S-wave).

On-mass shell: $F_l^{on} = F_l(p_0=0, p=p_s; p_s) = F(p_s)$
– physical amplitude (determines phase shifts).

Equation contains singularities.

Solution: BS amplitude is also singular.

The main problem is to treat singularities
(analytically or numerically).

To find binding energy, we can make Wick rotation in the BS equation and calculate BS amplitude in Euclidean space.

But we need it in Minkowski space!

• Two methods in Minkowski space

- Via Nakanishi representation
- Direct and accurate treating of singularities

• Minkowski space solution

for bound states

via Nakanishi representation

K. Kusaka, A.G. Williams, Phys.Rev. **D51** *(1995) 7026;*
V.A. Karmanov, J. Carbonell, Eur. Phys. J. A **27**, *1 (2006)*

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{-ig(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\ z' - i\epsilon\right]^3}.$$

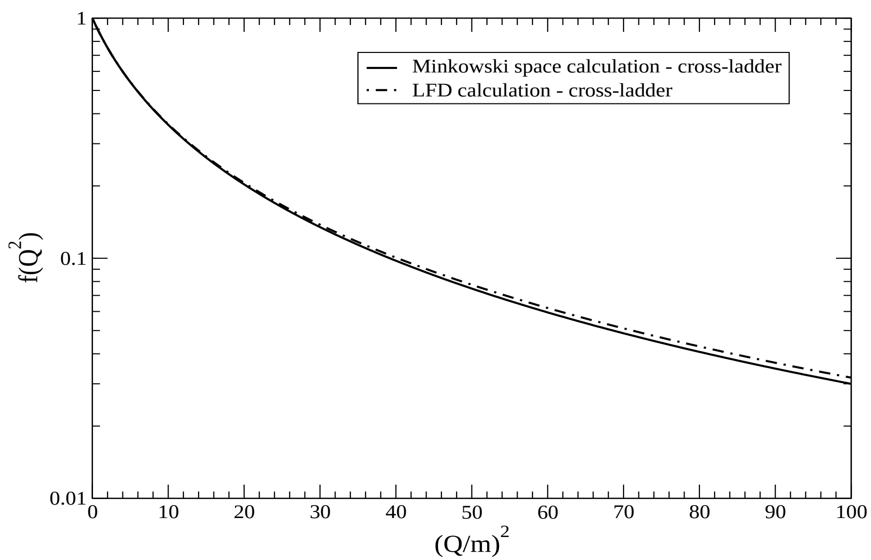
$\Phi(k,p)$ is singular.
$g(\gamma',z')$ is not singular.

From the BS equation one derives equation for $g(\gamma',z')$.

• Elastic e.m. form factor

It is analytically expressed via $g(\gamma, z)$ and easily calculated.

J. Carbonell, V.A. Karmanov, M. Mangin-Brinet,
Eur. Phys. J. **A 39** *(2009) 53-60*



Form factor via Minkowski BS amplitude (solid curve), and in LFD (dot-dashed curve).

• Scattering states

via Nakanishi representation

For the scattering states, an analogous formalism (Nakanishi) was developed in

T. Frederico, G. Salmè, and M. Viviani,

Phys. Rev. D 85 (2012) 036009.

The numerical solution was not yet obtained (as far as I know).
Though, there is no any principle obstacles.

For the present, this approach (Nakanishi) cannot be yet applied to calculating transition form factor.

• Another method

to find the bound and scattering states

in Minkowski space

Direct and accurate treating of singularities

V.A. Karmanov, J. Carbonell,

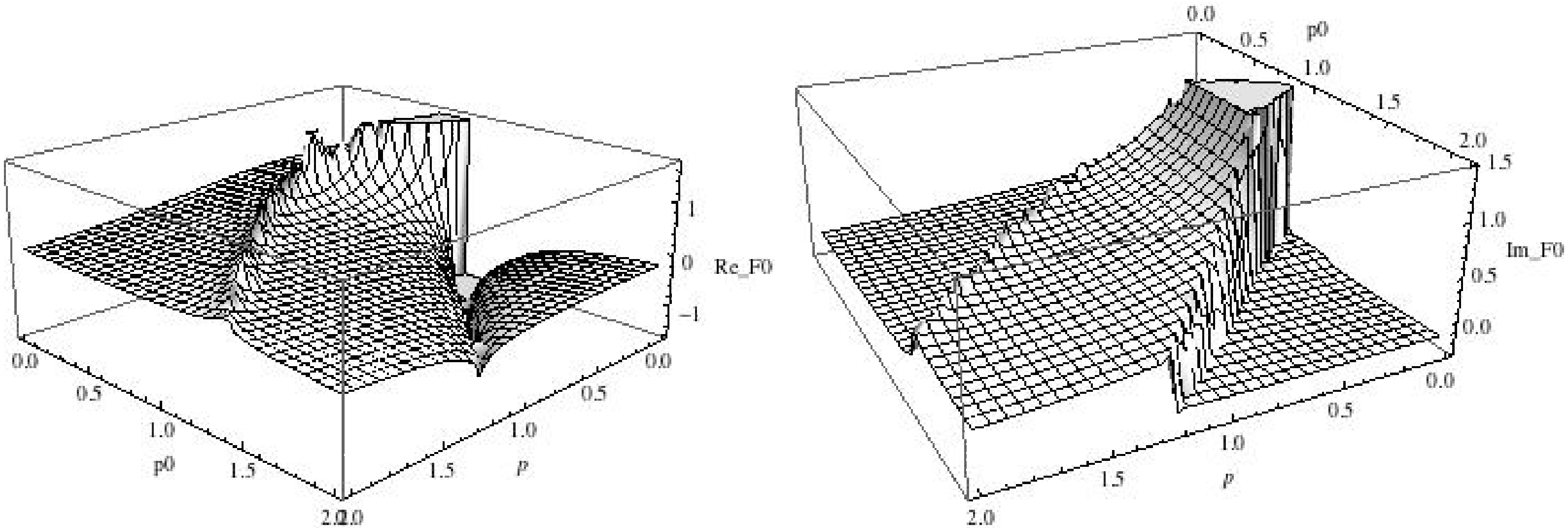
FB20, Fukuoka, Japan, Aug. 2012;

Few-Body Syst., **54***, 1509 (2013).*

(example of method:
calculating transition form factor, this talk)

• Scattering state solution

Minkowski space half-off-shell amplitude
found in our work for the first time



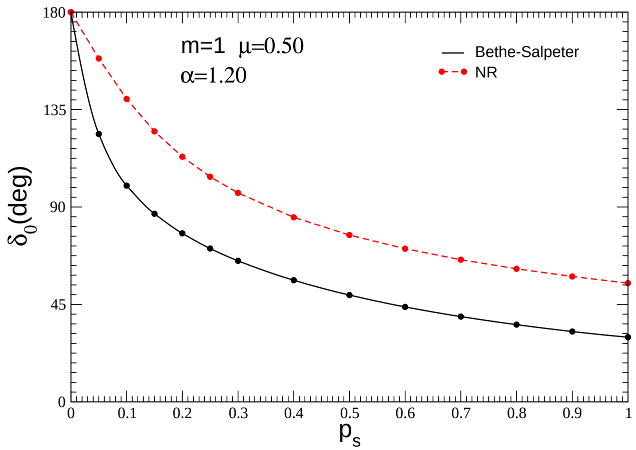
Real (left panel) and imaginary (right panel) parts
of the off-shell amplitude $F(p_0, p; p_s)$.

• Phase shift

On-shell amplitude – phase shift

Particular case: $F_l^{on} = F_l(p_0 = 0, p = p_s; p_s)$

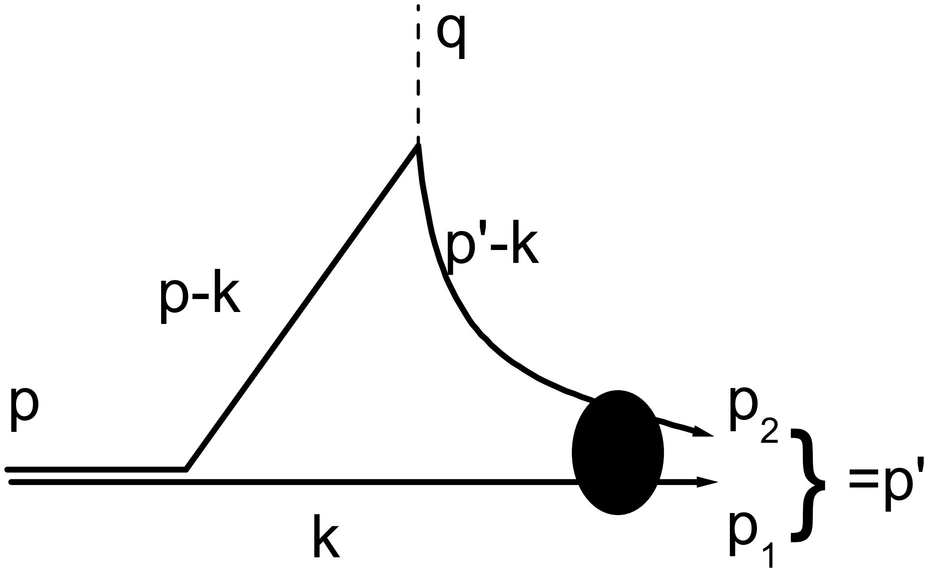
$$S_l = e^{i2\delta_l} = 1 + \frac{2ip_s F_l^{on}}{\varepsilon_{p_s}}$$



Phase shift (degrees) calculated via BS equation (solid black) compared to the non-relativistic results (dashed red).

Coincides with one found via BS 40 years ago: Tjon et al.

• Transition amplitude



Feynman diagram for the EM form factor.

$$\tilde{J}_\mu = i \int \frac{d^4k}{(2\pi)^4} \frac{(p_\mu + p'_\mu - 2k_\mu)\Gamma_i\left(\frac{1}{2}p - k, p\right)\Gamma_f\left(\frac{1}{2}p' - k, p'\right)}{(k^2 - m^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon][(p'-k)^2 - m^2 + i\epsilon]},$$

• Restoring gauge invariance

$$q\cdot\tilde{J} \neq 0 \quad \rightarrow \quad J_\mu = \tilde{J}_\mu - \frac{q_\mu}{q^2}(q\cdot\tilde{J}), \quad q\cdot J = 0$$

Unique decomposition:

$$J_\mu = \left[(p_\mu + p'_\mu) + \frac{({M'}^2 - M^2)}{Q^2}(p'_\mu - p_\mu)\right] F(Q^2)$$

In the frame where $p_0 = p'_0$ (but $|\vec{p}| \neq |\vec{p'}|$!):

$$J_0 = 2p_0 F(Q^2)$$

Calculating J_0, we find form factor.

• Treating pole singularities

$$
\begin{aligned}
F &= \int \frac{\dots d^4k}{(k^2 - m^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon][(p'-k)^2 - m^2 + i\epsilon]} \\
&= \int [PV + \delta][PV + \delta][PV + \delta] dk_0 d^3k \\
&= \int (f_3 + f_2 + f_1) dk_0 d^3k
\end{aligned}
$$

$$
\begin{aligned}
f_3 &= PV \cdot PV \cdot PV \\
f_2 &= PV \cdot PV \cdot \delta \\
f_1 &= PV \cdot \delta \cdot \delta
\end{aligned}
$$

• Treating pole singularities in f_3

$f_3 = PV \cdot PV \cdot PV \quad \leftarrow$ 3 propagators = 6 poles

$$f_3 = (f_3 - h_3) + h_3$$

where

$$\begin{aligned} h_3(k_0) &= \frac{g_1}{k_0 - E_{\vec{k}}} + \frac{g_2}{k_0 + E_{\vec{k}}} + \frac{g_3}{k_0 - p_0 - E_{\vec{p}-\vec{k}}} \\ &+ \frac{g_4}{k_0 - p_0 + E_{\vec{p}-\vec{k}}} + \frac{g_5}{k_0 - p_0' - E_{\vec{p}'-\vec{k}}} + \frac{g_6}{k_0 - p_0' + E_{\vec{p}'-\vec{k}}} \end{aligned}$$

g_{1-6} do not depend on k_0.
From the condition that $(f_3 - h_3)$ is not singular we find g_{1-6}.
Then $(f_3 - h_3)$ is easily integrated numerically,
whereas the singular integral $PV \int h_3 dk_0$ is calculated analytically.

• Integrating h_3

$$\begin{aligned}
F_{3,fv}(z,k) &= PV\int_{-L}^{L} h_3(k_0)dk_0 \\
&= \ldots \log\frac{L-E_{\vec{k}}}{L+E_{\vec{k}}} && (1)\\
&+ \ldots \log\frac{L+E_{\vec{k}}}{L-E_{\vec{k}}} && (2)\\
&+ \ldots \log\frac{L-p_0-E_{\vec{p}-\vec{k}}}{L+p_0+E_{\vec{p}-\vec{k}}} && (3)\\
&+ \ldots \log\frac{L-p_0+E_{\vec{p}-\vec{k}}}{L+p_0-E_{\vec{p}-\vec{k}}} && (4)\\
&+ \ldots \log\frac{L-p_0-E_{\vec{p'}-\vec{k}}}{L+p_0+E_{\vec{p'}-\vec{k}}} && (5)\\
&+ \ldots \log\frac{L-p_0+E_{\vec{p'}-\vec{k}}}{L+p_0-E_{\vec{p'}-\vec{k}}} && (6)
\end{aligned}$$

$\log$-singularities, can be integrated over d^3k numerically.

• Treating f_2 and f_1

$f_2 = PV \cdot PV \cdot \delta$: is integrated over dk_0 by means of the delta-function. Still needs one subtraction. Double integral over $d^3k = 2\pi k^2 dk dz$ is numerical.

$f_1 = PV \cdot \delta \cdot \delta$: is integrated over dk_0 and dz by means of two the delta-functions.
Does not need any subtraction.
Single integral over dk is numerical.

In this way we find the transition form factor $F(Q^2)$

• Test via elastic form factor

$$M' = M$$

$$(p+p')^\nu F_{el}(Q^2) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{(p+p'-2k)^\nu}{(k^2-m^2+i\epsilon)}\,\frac{\Gamma\left(\frac{1}{2}p-k,p\right)\Gamma\left(\frac{1}{2}p'-k,p'\right)}{[(p-k)^2-m^2+i\epsilon][(p'-k)^2-m^2+i\epsilon]},$$

Put $\Gamma = 1$. Use the Feynman parametrization:

$$\frac{1}{abc} = \int_0^1 du \int_0^{1-u} \frac{2dv}{(au+bv+c(1-u-v))^3}$$

Get:

$$F_{el}(Q^2) = \frac{1}{16\pi^2}\int_0^1 du \int_0^{1-u} \frac{(1-u-v)dv}{[m^2-(1-u-v)(u+v)M^2+uvQ^2]}$$

• Numerical test

Take, for example, $m = 1$, $M = 1.9$, $Q^2 = 10$. We find:

$$F_{el}(Q^2) = 0.00200374$$

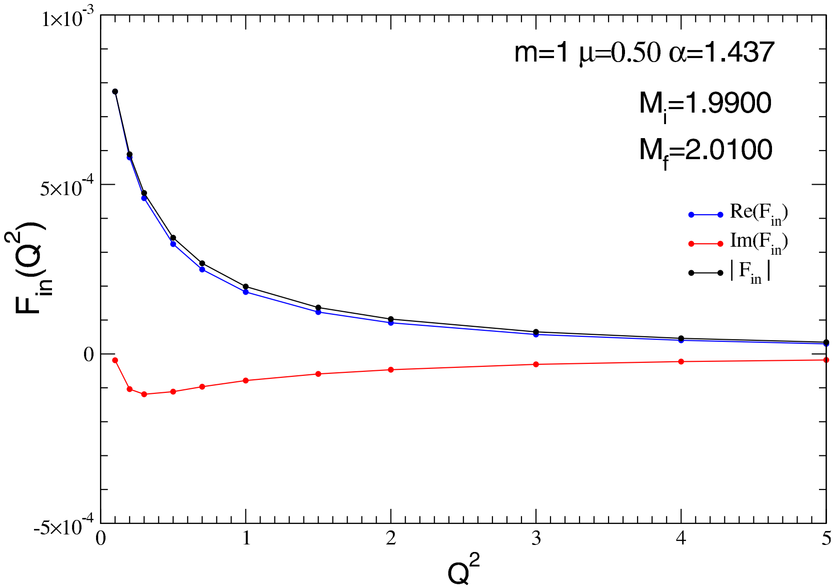
Exactly the same value is obtained by the method developed
to calculate the transition form factor!

Form factor is correct if the Bethe-Salpeter amplitudes
Γ_i, Γ_f are correct.

The Bethe-Salpeter amplitudes Γ_i, Γ_f passed
via multiple tests in our previous work.

• Transition form factor

Numerical result



Real (blue), imaginary (red) parts and module (black) of the transition form factor $F_{in}(Q^2)$.

• Conclusion

- To calculate form factors (elastic and inelastic) we need BS amplitudes in Minkowski space.
- These solutions are found (for bound and scattering states).
- Inelastic form factor (for the transition: bound $\to$ scattering states) is expressed in terms of the initial (bound state) and final (scattering state) BS amplitudes.
- The singularities are properly treated, so that no problem with numerical calculations.
- By this method, the transition form factor is calculated, the results is found.
- Calculations are done for the spineless case, but the method is applicable to realistic systems.

Thank you!