Multipole Analysis of Radio-Frequency Reactions in Ultracold Atoms

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האוניברסיטה העברית בירושלים
The Hebrew University of Jerusalem

Outline

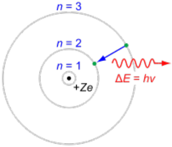
References:

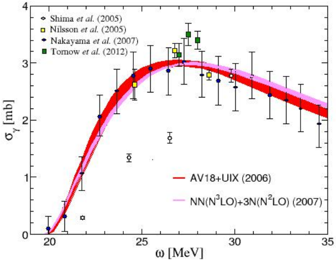
- E. Liverts, B. Bazak, and N. Barnea, Phys. Rev. Lett. **108**, 112501 (2012)
- B. Bazak, E. Liverts, and N. Barnea, Phys. Rev. A **86**, 043611 (2012)
- B. Bazak, E. Liverts, and N. Barnea, Few Body Syst. **54**, 667 (2013)
- B. Bazak and N. Barnea, arXiv:1305.4368 [cond-mat.quant-gas]

Outline

What Can We Learn From Photo Reactions?

1. Understanding of the systems at hand.
2. A test of the Hamiltonian at regimes not accessible by elastic reactions.
3. Reaction rates as input for experiments or applications (e.g. astrophysics).
4. Underlying degrees of freedom.
5. The transition from single particle to collective behavior.





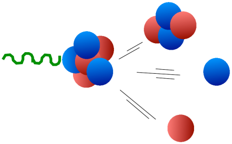
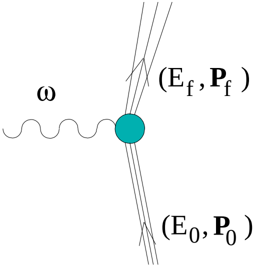


Photo Reactions

The interaction Hamiltonian between the photon field $\boldsymbol{A}(\boldsymbol{x})$ and the atomic/nuclear system

$$H_I = -\frac{e}{c}\int d\boldsymbol{x}\boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{J}(\boldsymbol{x})$$



The current is a sum of convection and spin currents

$$\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{J}_c(\boldsymbol{x}) + \nabla \times \boldsymbol{\mu}(\boldsymbol{x})$$

$$H_I = -\frac{e}{c}\int d\boldsymbol{x}\left\{\boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{J}_c(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x}) \cdot \boldsymbol{\mu}(\boldsymbol{x})\right\}$$

- Classically, the convection current $J_c = \sum_i Q_i v_i$ is the flow of the charged particles.
- In nuclear physics, the convection current is dominant at low energies.
- Ultracold atoms are neutral $J_c(x) = 0$ and the current $\mu(x)$ is dominated by the spins.

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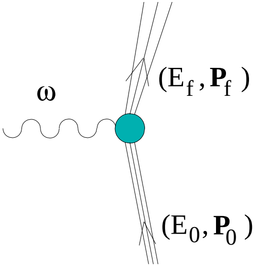

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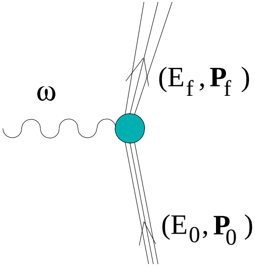

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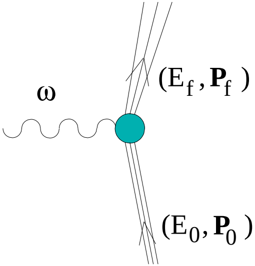

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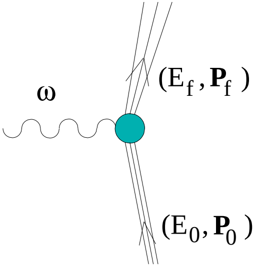

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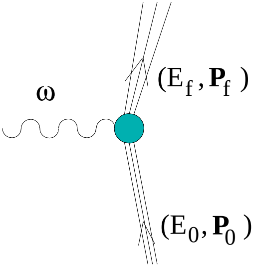

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Efimov Physics and Universality

- *Borromean regime*: A 3-body bound state exists even when the 2-body system is unbound.
- In nuclear physics, ^{6}He is bound while ^{5}He, *n*-*n* - not.

- *The unitary limit*: $E_2 = 0$, $a_s \longrightarrow \infty$.
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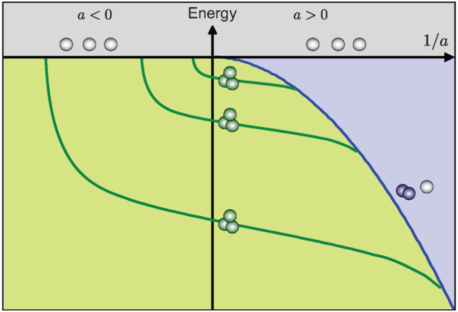

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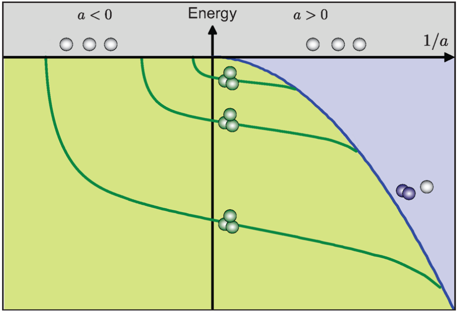

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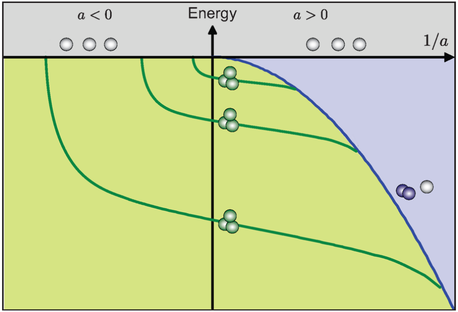

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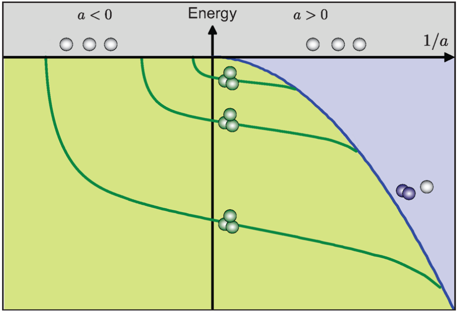

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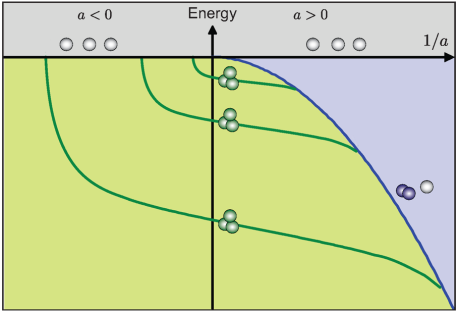

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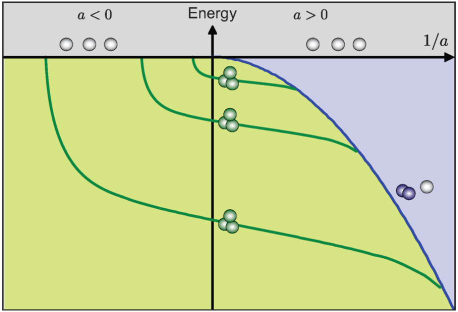

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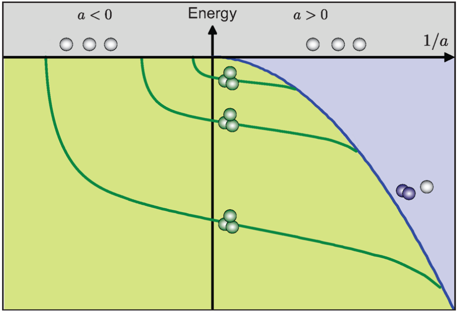

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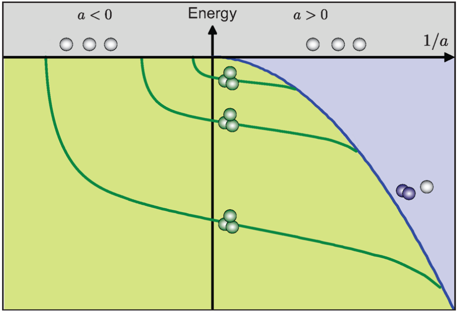

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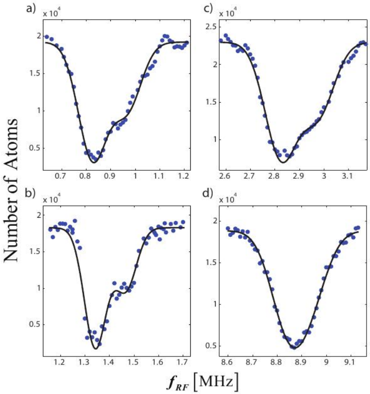
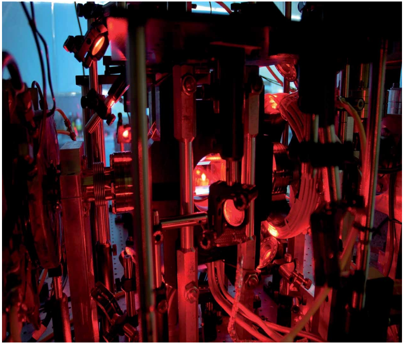
RF-induce atom loss resonances for different values of bias magnetic fields.





RF association of ^{7}Li dimers and trimers at 1.5 μK

O. Machtey, Z. Shotan, N. Gross and L. Khaykovich, Phys. Rev. Lett. **108**, 210406 (2012)

Outline

The Static Response - Inelastic Reactions

- The response of an A-particle system is closely related to the static moments of the charge density

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{A} Q_i \delta(\boldsymbol{x} - \boldsymbol{r}_i)$$

- The Fourier Transform

$$\rho(\boldsymbol{k}) = \int d\boldsymbol{x} \rho(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \sum_{i=1}^{A} Q_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}$$

- In the long wavelength limit $k \longrightarrow 0$

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- **Conclusion A:** In general the Dipole is the leading term.
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Photo Reactions with Ultracold Atoms

- For RF photons in the few MHz region the wave length is meters so $kR \ll 1$.
- The atoms reside in a strong magnetic field, with well defined m_F,

$$|\Psi_0\rangle = \Phi_0(\boldsymbol{r}_i)|m_F^1 m_F^2 \dots m_F^A\rangle$$

- In the final state the photon can either change one of the spins or leave them untouched.
- Spin-flip reaction

$$\boxed{|m_F^1 m_F^2 \dots m_F^A\rangle \longrightarrow |m_F^1 m_F^2 \pm 1 \dots m_F^A\rangle}$$

- Frozen-Spin reaction

$$\boxed{|m_F^1 m_F^2 \dots m_F^A\rangle \longrightarrow |m_F^1 m_F^2 \dots m_F^A\rangle}$$

N. Gross and L. Khaykovich, Phys. Rev. **A** 77, 023604 (2008)

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- For RF photons in the few MHz region the wave length is meters so $kR \ll 1$.
- The atoms reside in a strong magnetic field, with well defined m_F,

$$|\Psi_0\rangle = \Phi_0(\boldsymbol{r}_i)|m_F^1 m_F^2 \ldots m_F^A\rangle$$

- In the final state the photon can either change one of the spins or leave them untouched.
- Spin-flip reaction

$$|m_F^1 m_F^2 \ldots m_F^A\rangle \longrightarrow |m_F^1 m_F^2 \pm 1 \ldots m_F^A\rangle$$

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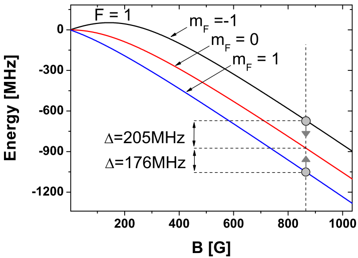

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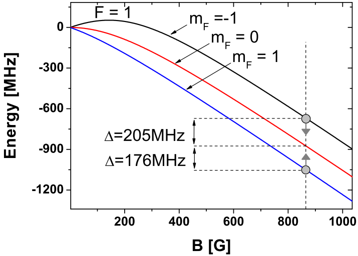

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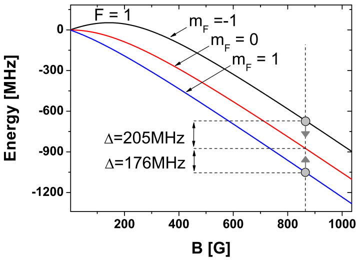

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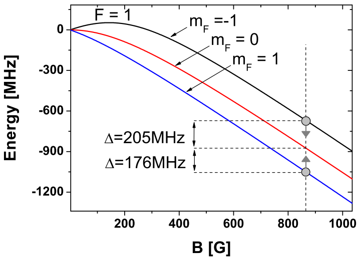

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$$R(\omega) = Ck \sum_{f,\lambda} \left| \langle \Phi_f | \Phi_0 \rangle \right|^2 \delta(E_f - E_0 - \omega)$$

- For Frozen-Spin reactions we get a sum of the monopole operator $\hat{M} = R^2 = \sum r_i^2$ and the Quadrupole operator $\hat{Q} = \sum r_i^2 Y_2(\hat{r}_i)$

$$O = \alpha \hat{M} + \beta \hat{Q}$$

- The response is given by

$$R(\omega) = k^5 \sum_{f,\lambda} \left| \langle \Phi_f | O | \Phi_0 \rangle \right|^2 \delta(E_f - E_0 - \omega)$$

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Outline

Photoassociation of the Atomic Dimer

- For the dimer case, the response function can be written as

$$R(\omega) = C\omega^5 \left[\frac{1}{6^2} |\langle \psi_0 \| \hat{M} \| \varphi_0(q) \rangle|^2 + \frac{1}{5 \cdot 15^2} |\langle \psi_0 \| \hat{Q} \| \varphi_2(q) \rangle|^2 \right]$$

- The bound state wave function is

$$\psi_0 = Y_0 \sqrt{2\kappa} e^{-\kappa r}/r \quad ; \quad \kappa \approx 1/a_s$$

- The continuum state wave function is

$$\varphi_{lm}(q) = Y_{lm}(\hat{r}) 2q [\cos \delta_l j_l(qr) - \sin \delta_l n_l(qr)]$$

- The $l = 0$ matrix element

$$|\langle \psi_0 \| \hat{M} \| \varphi_0(q) \rangle|^2 = \frac{1}{4\pi} \left(\frac{4q\sqrt{2\kappa}}{(q^2+\kappa^2)^3} \right)^2 \left[\cos \delta_0 (3\kappa^2 - q^2) - \sin \delta_0 \frac{\kappa}{q} (3q^2 - \kappa^2) \right]^2$$

- The $\ell = 2$ matrix element, assuming $\delta_2 = 0$

$$|\langle \psi_0 \| \hat{Q} \| \varphi_2(q) \rangle|^2 = \frac{5}{4\pi} \left[\frac{16q^3\sqrt{2\kappa}}{(q^2+\kappa^2)^3} \right]^2$$

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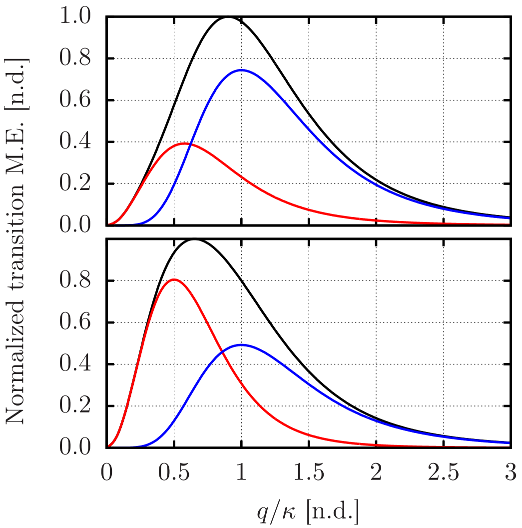
The s-wave and d-wave components in the response function

- upper panel $a/r_{eff} = 2$
- lower panel $a/r_{eff} = 200$
- red - r^2 monopole
- blue - quadrupole
- black - their sum



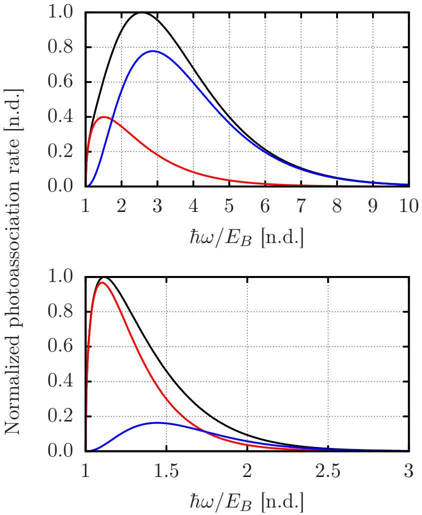
Dimer Photoassociation Rates

Photoassociation of ^{7}Li dimers

$a_s = 1000a_0$
$T = 25\mu K$ (upper panel)
$T = 5\mu K$ (lower panel)

red - r^2 monopole, blue - quadrupole, black - sum

The relative contribution to the peak



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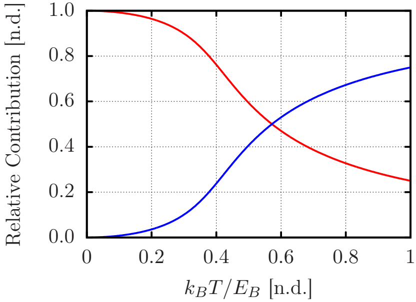

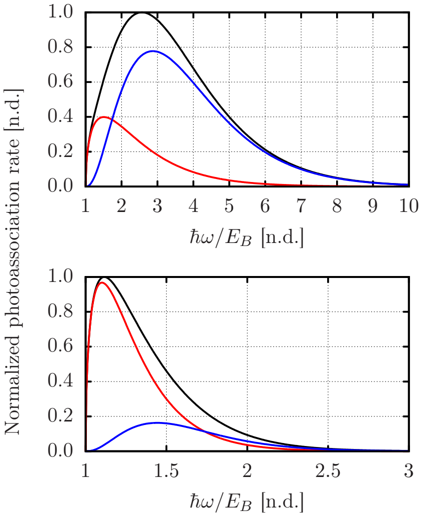

Outline

Road-map for Efimov Physics

To get analytic results for the 3-body problem,

- Assume short-range interaction and large scattering length
- Remove center of mass and adopt the hyper-spherical coordinates

$$(r_1, r_2, r_3) \to (R_{CM}, x_i, y_i) \to (R_{CM}, \rho, \alpha_i, \hat{x}_i, \hat{y}_i)$$

- Use the adiabatic expansion (Born-Oppenheimer like), where ρ is the slow coordinate

$$\Psi(\rho, \Omega) = \sum_n \rho^{-5/2} f_n(\rho) \Phi_n(\rho, \Omega)$$

- Decompose into Faddeev amplitudes to impose symmetry and boundary condition

$$\Phi_n(\rho, \Omega) = \sum_i \phi_{n,i}(\rho, \Omega_i)$$

- For given ρ, solve the hyper-angular equation,

$$\left(\hat{K}^2 + \frac{2m}{\hbar^2} \rho^2 \sum_i V(\sqrt{2}\rho \sin \alpha_i) + 4 \right) \Phi_n(\rho, \Omega) = \nu_n^2 \Phi_n(\rho, \Omega).$$

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- Therefore, the potential is expressed as boundary condition,

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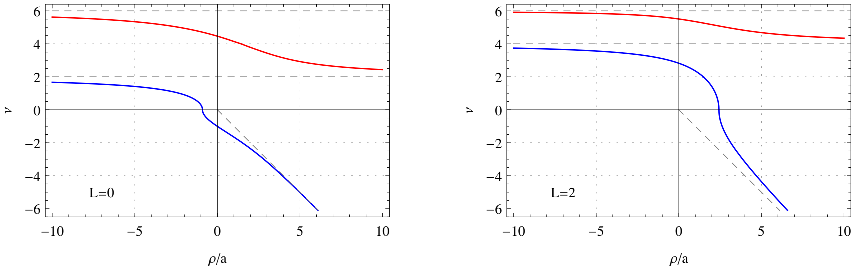

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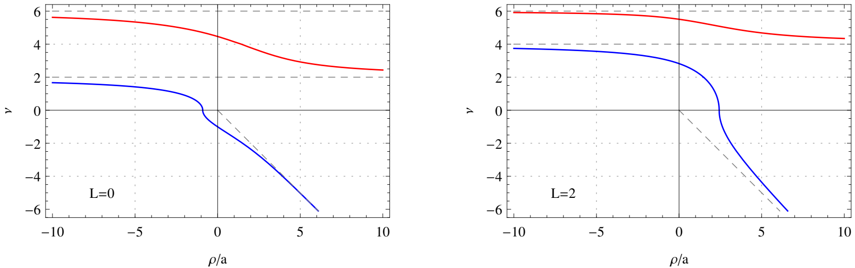

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- The result is a 1-D equation for $f(\rho)$ and E, with an effective $\frac{1}{\rho^2}$ potential,

$$\left(-\frac{\partial^2}{\partial\rho^2} + \frac{2m}{\hbar^2}\left(\frac{\hbar^2}{2m}\frac{\nu_n^2(\rho) - 1/4}{\rho^2} - Q_{nn} - E\right)\right)f_n(\rho) = \sum_{n\neq n'}(2P_{nn'}\frac{\partial}{\partial\rho} + Q_{nn'})f_{n'}(\rho)$$

The Unitary Limit

- In the unitary limit, $|a| \to \infty$, ν does not depend on ρ, and the channels decouples.
- The hyper-radial equation is similar to the Bessel equation,

$$-\frac{d^2 f(\rho)}{d\rho^2} + \frac{\nu_L^2 - 1/4}{\rho^2} f(\rho) = \epsilon f(\rho)$$

with $\nu_0 \approx 1.00624i$, and $\nu_2 \approx 2.82334$.

1. Bound state, $E_n = -\hbar^2 \kappa_n^2 / 2m < 0$:

$$f_B^{(n)}(\rho) \propto \kappa_n \sqrt{\rho} K_{\nu_0}(\kappa_n \rho)$$

where to avoid the Thomas collapse, a 3-body repulsive force is to be introduced, for example $U(\rho \leq \rho_0) = \infty$ for some finite ρ_0, resulting in the famous Efimov spectrum,

$$\frac{E_n}{E_0} = e^{-2\pi n / |\nu_0|} \approx 515^{-n}.$$

2. Scattering state, $E = \hbar^2 q^2 / 2m > 0$:

$$f_L(\rho) \propto \sqrt{\frac{q\rho}{R}} \left[\sin \delta_L J_{\nu_L}(q\rho) + \cos \delta_L Y_{\nu_L}(q\rho) \right]$$

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- In the unitary limit, $|a| \to \infty$, ν does not depend on ρ, and the channels decouples.
- The hyper-radial equation is similar to the Bessel equation,

$$-\frac{d^2 f(\rho)}{d\rho^2} + \frac{\nu_L^2 - 1/4}{\rho^2} f(\rho) = \epsilon f(\rho)$$

with $\nu_0 \approx 1.00624i$, and $\nu_2 \approx 2.82334$.

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$$f_B^{(n)}(\rho) \propto \kappa_n \sqrt{\rho} K_{\nu_0}(\kappa_n \rho)$$

where to avoid the Thomas collapse, a 3-body repulsive force is to be introduced, for example $U(\rho \le \rho_0) = \infty$ for some finite ρ_0, resulting in the famous Efimov spectrum,

$$\frac{E_n}{E_0} = e^{-2\pi n / |\nu_0|} \approx 515^{-n}.$$

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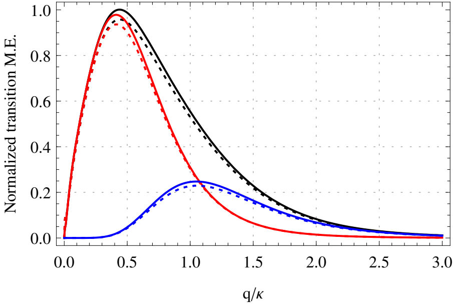

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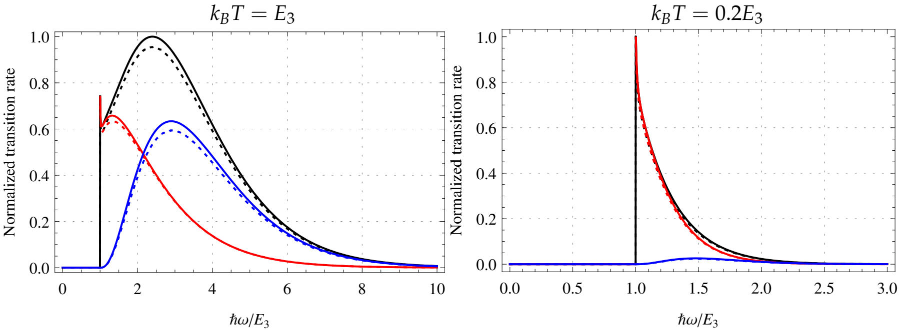
Trimer Photoassociation: Results



red - r^2 monopole, blue - quadrupole, and black - their sum

Log-periodic oscillations

- We found another fingerprint of the Efimov physics, a *log-periodic* oscillations:
- For the s-wave, near threshold,

$$I \approx 1 + \frac{B_2}{2}\cos(2s_0 \ln q/\kappa), \quad B_2 \approx 8.5\%$$

- For any multipole, at the high-frequency tail
- The near threshold oscillations may be blurred by the finite energy width of the trimer.
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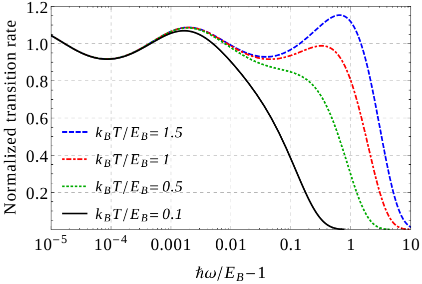

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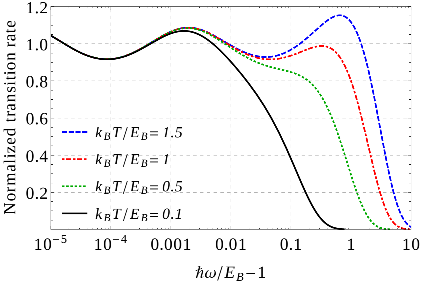

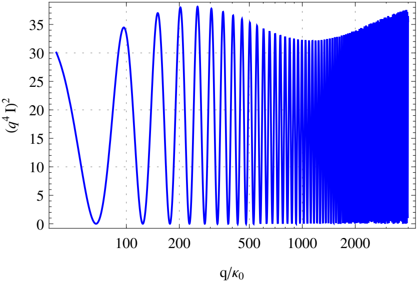

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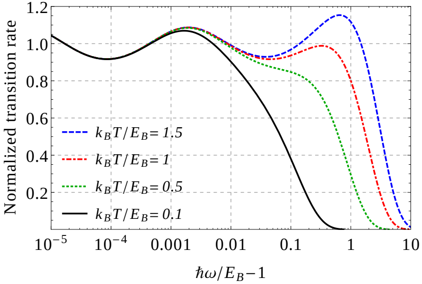

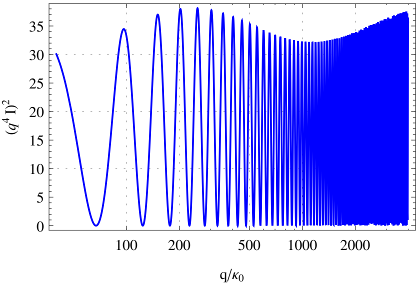

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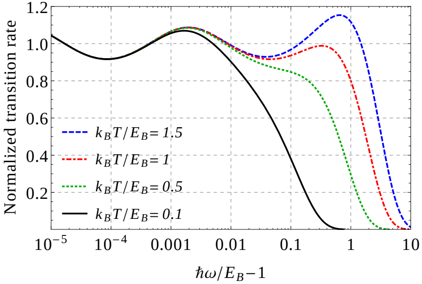

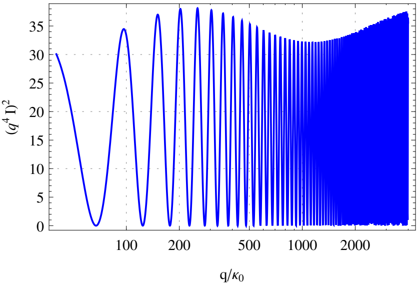

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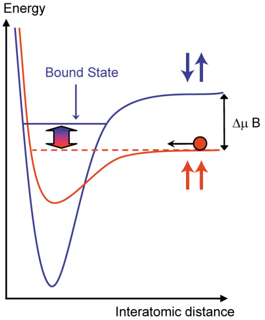
Experimental realization

- Magnetically Feshbach resonance based on the spin dependence of the molecular interaction.



from Ketterle group site

- Therefore, m_f ceases to be good quantum number, but $\sum m_f$ still is.
- For example, the state $|11\rangle$ is mixed with $|02\rangle$ and $|20\rangle$
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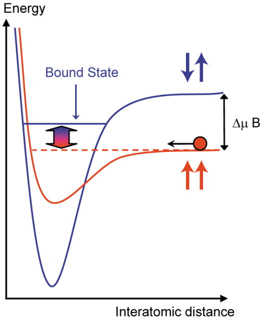

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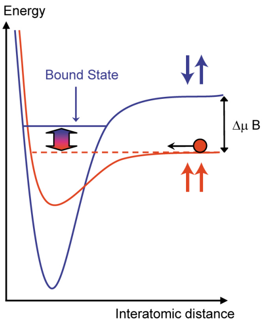

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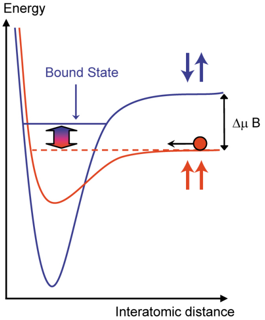

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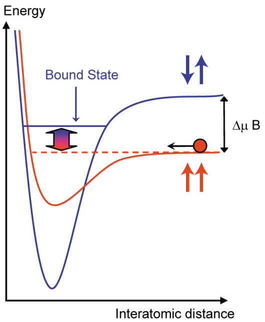

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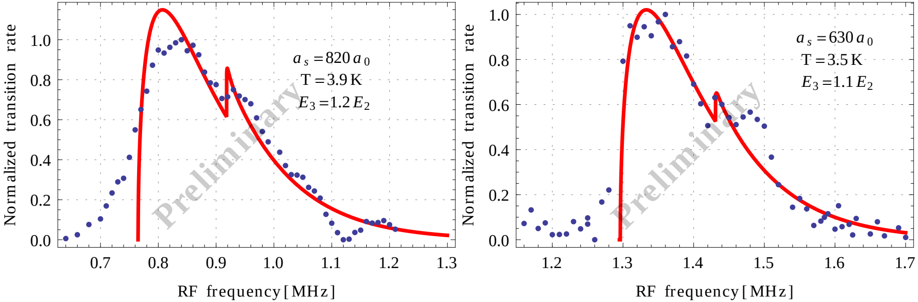
- Putting all together, fitting our model to the Khaykovich group data:



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2. For spin-flip reaction, the Franck-Condon factor is the leading contribution to the cross-section, and $R(\omega) \propto \omega$.
3. For frozen-spin reactions the monopole R^2 and the quadrupole are the leading terms, and $R(\omega) \propto \omega^5$.
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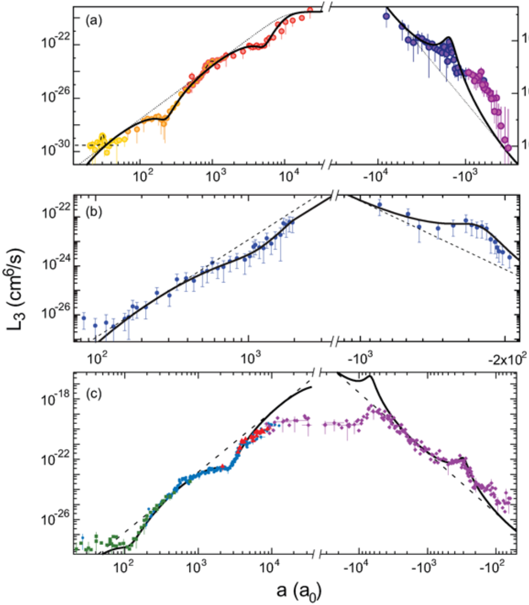
BACKUP slides

Efimov Physics in Ultracold Atoms



F. Ferlaino and R. Grimm, Physics **3**, 9 (2010)

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  Nature Phys. **5**, 586 (2009).
- ^{7}Li
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  Phys. Rev. Lett. **103**, 163202 (2009).
- ^{7}Li
  S.E. Pollack, D. Dries, and R.G. Hulet,
  Science **326**, 1683 (2009).

The Hyper-Spherical Coordinates

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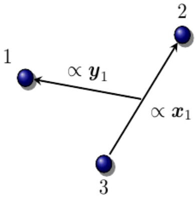

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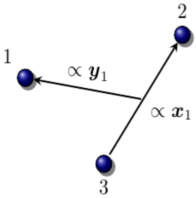

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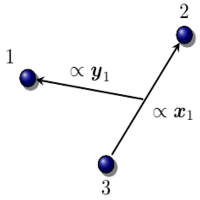

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The Adiabatic Expansion

- Next we apply the adiabatic expansion,

$$\Psi(\rho,\Omega) = \sum_n \rho^{-5/2} f_n(\rho)\Phi_n(\rho,\Omega),$$

- $\Phi_n(\rho,\Omega)$ is the solution of the hyper angular equation corresponding to the eigenvalue ν_n^2,

$$\left(\hat{K}^2 + \frac{2m}{\hbar^2}\rho^2 \sum_i V(\sqrt{2}\rho \sin\alpha_i) + 4\right)\Phi_n(\rho,\Omega) = \nu_n^2 \Phi_n(\rho,\Omega).$$

- $f_n(\rho)$ is the solution of the hyper-radial equation,

$$\left(-\frac{\partial^2}{\partial\rho^2} + \frac{2m}{\hbar^2}(V_{\mathrm{eff}}(\rho) - E)\right) f_n(\rho) = \sum_{n \neq n'} (2P_{nn'}\frac{\partial}{\partial\rho} + Q_{nn'}) f_{n'}(\rho)$$

- where the effective potential is

$$V_{\mathrm{eff}}(\rho) = \frac{\hbar^2}{2m}\frac{\nu_n^2(\rho) - 1/4}{\rho^2} - Q_{nn}$$

and the non-adiabatic couplings are

$$P_{nn'}(\rho) = \left\langle \Phi_n(\rho,\Omega) \middle| \frac{\partial}{\partial\rho} \middle| \Phi_{n'}(\rho,\Omega) \right\rangle_\Omega$$

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The Adiabatic Expansion

- Next we apply the adiabatic expansion,

$$\Psi(\rho,\Omega) = \sum_n \rho^{-5/2} f_n(\rho)\Phi_n(\rho,\Omega),$$

- $\Phi_n(\rho,\Omega)$ is the solution of the hyper angular equation corresponding to the eigenvalue ν_n^2,

$$\left(\hat{K}^2 + \frac{2m}{\hbar^2}\rho^2 \sum_i V(\sqrt{2}\rho\sin\alpha_i) + 4\right)\Phi_n(\rho,\Omega) = \nu_n^2 \Phi_n(\rho,\Omega).$$

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The Faddeev Decomposition

- Using Faddeev decomposition,

$$\Phi_n(\rho,\Omega) = \sum_i \phi_{n,i}(\rho,\Omega_i)$$

- We assume our interaction is of *zero range* and *s-wave* only, Therefore the only partial wave to be considered for the bound state is $l_x = 0, l_y = L$.
- Now the solution is,

$$\phi_{n,i}(\rho,\Omega_i) = \frac{g_{\nu,L}(\alpha_i)}{\sin(2\alpha_i)} Y^{L,M}_{l_x,l_y}(\hat{x}_i,\hat{y}_i)$$

where

$$g_{\nu,L}(\alpha_i) = \cos^L\alpha\left(\frac{\partial}{\partial\alpha}\frac{1}{\cos\alpha}\right)^L \sin\left[\nu\left(\alpha-\frac{\pi}{2}\right)\right],$$

$$Y^{L,M}_{l_x,l_y}(\hat{x},\hat{y}) = \sum_{m_x,m_y}\langle l_x m_x l_y m_y|LM\rangle Y^{m_x}_{l_x}(\hat{x})Y^{m_y}_{l_y}(\hat{y})$$

- In the low energy limit, the boundary condition reads

$$\left[\frac{1}{2\alpha_i\Phi}\frac{\partial}{\partial\alpha_i}2\alpha_i\Phi\right]_{\alpha_i=0} = -\sqrt{2}\rho\frac{1}{a_s}$$

A. Cobis, D.V. Fedorov, and A.S. Jensen, Phys. Rev. Lett. **79**, 2411 (1997).

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