BOUND STATE PROBLEM FOR THE THREE-BODY SCHRÖDINGER EQUATION WITH EUCLIDEAN INVARIANT DECAYING POTENTIAL

RYTIS JURŠĖNAS

Institute of Theoretical Physics and Astronomy of Vilnius University, A. Goštauto 12, LT01108, Vilnius, Lithuania

The problem — The main goal of the present work is to demonstrate a new technique for solving the bound state problem for the three-body Schrödinger equation [1].

The Schrödinger operator, $T+V$, contains the kinetic energy part T and the potential V. The reference space is a separable Hilbert space $\mathrm{H}_0 \equiv L^2(\mathbb{R}^9)$. Due to the isomorphism $\mathrm{H}_0 \cong \mathrm{H}\otimes\mathrm{H}\otimes\mathrm{H}$, where $\mathrm{H} \equiv L^2(\mathbb{R}^3)$, the kinetic energy part T: $\mathrm{D}(T) \to \mathrm{H}_0$ can be realized via the operator sum of $-(2m_i)^{-1}\Delta_i$ for all $i = 1,2,3$, provided the domain $\mathrm{D}(T)$ of T is a dense subset of H_0. The numbers $\{m_i > 0\colon\ i = 1,2,3\}$ are referred to as masses, the Laplacian Δ_i is in vectors $\mathbf{r}_i \in \mathbb{R}^3$, all $i = 1,2,3$.

The reference space H_0 implies that the potential V, with its domain of definition $\mathrm{D}(V)$, is defined on the spatial coordinates only. Moreover, we require that V be the operator of multiplication by the sum of $V(\mathbf{r}_{ij})$, all $1 \leq i < j \leq 3$, such that $\mathrm{D}(V) \supseteq \mathrm{D}(T)$. The present requirement allows us to apply the Kato–Rellich theorem to define a self-adjoint operator $T+V$ on $\mathrm{D}(T)$, provided T is positive definite and self-adjoint. Throughout, vectors $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ for all $1 \leq i < j \leq 3$. In addition, we restrict ourselves to the cases when the functions $\{V(\mathbf{r}_{ij}) \in \mathbb{R}\colon\ 1 \leq i < j \leq 3\}$ fulfill the conditions: (a) $V(\mathbf{r}_{ij}) \to 0$ as $r_{ij} \to 0$ for $1 \leq i < j \leq 3$, where r_{ij} denotes the absolute value of $\mathbf{r}_{ij}$; (b) $V(\mathbf{r}_{ij}) = V(-\mathbf{r}_{ij})$, $1 \leq i < j \leq 3$, so that V is invariant under Euclidean moves; (c) $V(\mathbf{r}_{ij})$ is of class $C^\infty(\mathbb{R}^3)$, with possible singularities at $\mathbf{r}_i = \mathbf{r}_j$ for $i \neq j$; (d) the left-hand side of the expression $\sum_{i<j} \partial^k V(\mathbf{r}_{ij})/\partial r_{ij}^k = 0$, for some $k = 1,2,\ldots$, can be reduced into the equation depending on only one variable; for example, if $V(\mathbf{r}_{ij}) = b_{ij} r_{ij}^{-a}$, $b_{ij} \in \mathbb{R}$, $a > 0$, then condition (d) obeys the form $b_{12} + \wp^k b_{23} + (\wp/c)^k b_{13} = 0$, with $k = a+1, a+2, \ldots$, which is dependent on variable $\wp = r_{12}/r_{23}$, here $c = (1 + \wp^2 + 2\wp\cos\theta)^{1/2}$, θ is the angle between unit vectors $\hat{\mathbf{r}}_{12}$ and $\hat{\mathbf{r}}_{23}$. It appears from the above condition that c can be represented in terms of one variable, $\wp$, only.

Under the preceding assumptions, one can prove that there exists a subspace of $\mathbb{R}^9$ such that $T+V$ projected onto it is $SO(3)$–invariant. This is the main statement of the problem. As an example, we show that for the Coulomb three-body problem, the $SO(3)$–invariant terms are just the radial Schrödinger operators in $L^2(0,\infty)$ with the potential of the form $A_\kappa r^{-\kappa-1}$, for $\kappa = 0,1,\ldots$; the coefficients $A_\kappa \in \mathbb{R}$ may be positive (repulsive case) or negative (attractive case), depending on κ. As a result, the present approach applied to the Coulomb problem puts on one stage both short-range and long-range potentials. In particular, condition (d) yields the stability criterion imposed on various mass (and charge) configurations as that considered by other authors; see [2,3] and the citations therein.

The method — We begin with $M = \{(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) \in \mathbb{R}^9\colon\ \mathbf{t}_1 + \mathbf{t}_2 = \mathbf{t}_3\}$, the nine-dimensional manifold containing six independent components. The configuration space of M is $SO(3)$.

Then M is a graph of the additive group $(\mathbb{R}^3;+)$ on $\mathbb{R}^6$. Based on this result one shows that $L^2(M) \equiv L^2(\mathbb{R}^6)$, with M endowed with (Lebesgue) measure. It follows from (a)–(d) that M can be represented by the equivalence classes of disjoint subsets $\{\bigcup_{\kappa=0}^{\infty} M_{\kappa+p}\colon\ p=0,1,\ldots\}$, where $M_\kappa = \{(\mathbf{r}_{12},\mathbf{r}_{23},\mathbf{r}_{13}) \in \mathbb{R}^9\colon\ \mathbf{r}_{12}+\mathbf{r}_{23}=\mathbf{r}_{13}, V_{\kappa+1}=0\}$. For $n=0,1,\ldots$, V_n denotes $G_z^n V$, where $G_z^n = G_z G_z \ldots G_z$ is applied to V n times, and G_z is the sum of gradients ∇_{ij} along z-axis with respect to r_{ij} for all $1 \leq i < j \leq 3$. As a result, $L^2(M)$ is found to be isomorphic with the equivalence classes (with respect to $p=0,1,\ldots$) of the direct sum of subspaces $L^2(M_{\kappa+p})$; the summation is performed over all $\kappa = 0,1,\ldots$ Elements G_z, V_0, V_1, …, V_κ form the nilpotent Lie algebra $\mathcal{A}(M_\kappa)$ on M_κ. As a result, one can construct the associated matrix Lie group $\mathcal{L}(M_\kappa) = \{e^{ita}\colon\ a \in \mathcal{A}(M_\kappa), t \in \mathbb{R}\}$ of matrices by introducing the representation ϱ of $\mathcal{A}(M_\kappa)$ on $\mathfrak{gl}(M_\kappa)$. Provided $\Pi\colon\ \mathcal{L}(M_\kappa) \to GL(M_\kappa)$ is a representation of $\mathcal{L}(M_\kappa)$ on $GL(M_\kappa)$, one finally derives the expression $\Pi(e^{\mathrm{i}a}) = e^{\mathrm{i}\varrho(a)}$, all $a \in \mathcal{A}$, which is, by default, applied to all $\varphi \in C^\infty(\mathbb{R}^6)$. The space $C^\infty(\mathbb{R}^6)$ is dense in $L^2(\mathbb{R}^6)$, and thus one can extend the above expression to the whole $L^2(M)$. One derives the equivalence class $\{\bigoplus_{\kappa=0}^{\infty} \mathrm{D}_{\kappa+p} \cong \mathrm{D}(T)\colon\ p=0,1,\ldots\}$, with $\mathrm{D}_s = \{\varphi \in \mathrm{D}_{0,s}\colon\ G\varphi = 0\}$ for all $s=\kappa,\kappa+1,\ldots$ Here $\mathrm{D}_{0,s}$ is the domain of $T+V_s$ in $L^2(M_s)$. Recalling that $\mathrm{D}(T+V)$ is a dense subset of H_0, and that $L^2(M)$ is a direct sum of $L^2(M_s)$, one deduces that the spectrum of $T+V$ is given by $\sum_{\kappa=0}^{\infty} \inf_p\{\mathrm{spec}(H_\kappa)\}$, where $H_\kappa = (T+V_\kappa) \restriction \mathrm{D}_\kappa$, and the infimum is taken over all $p=0,1,\ldots$ so that the element from $\{V_{\kappa+p+1}=0\colon\ p=0,1,\ldots\}$ yields the minimal values in the spectrum $\mathrm{spec}(H_\kappa)$ for a given $\kappa=0,1,\ldots$ To this end, the operator H_κ can be represented in three equivalent forms, eg $H_\kappa = H_\kappa^0 + \gamma(\nabla_{12}\cdot\nabla_{23})$, where $\gamma = m_2^{-1}$ and $H_\kappa^0 = T_1 + T_2 + V_\kappa$, where $T_1 = -\alpha\Delta_{12}$, $T_2 = -\beta\Delta_{23}$; parameters $\alpha = (2m_2)^{-1} + (2m_3)^{-1}$ and $\beta = (2m_1)^{-1} + (2m_2)^{-1}$. By [4], $\inf\mathrm{spec}(H_\kappa) = \inf\mathrm{spec}(H_\kappa^0)$ which means that the cross-term does not influence the ground state of H_κ.

Coulomb case — The application of the present method to the Coulomb three-body problem yields the discrete spectrum of H_κ^0, with $\kappa=0$,

$$\mathrm{spec}_d(H_0^0) = \left\{ -\frac{1}{4}\left(\frac{Z_1(Z_2+Z_3)}{n_1\sqrt{\alpha}} + \frac{Z_2Z_3}{n_2\sqrt{\beta}}\right)^2\colon\ \frac{1}{2} \leq \frac{n_1}{n_2}\sqrt{\frac{\alpha}{\beta}} \leq 1 \right\} \tag{1}$$

$(n_i = l_i+1, l_i+2,\ldots; i=1,2)$, where $\mathrm{spec}_d(H_0^0)$ also obeys an equivalent representation for $(n_1/n_2)\sqrt{\alpha/\beta} > 1$. The condition (d) infers $Z_1Z_2 + (n_1/n_2)\sqrt{\alpha/\beta}Z_2Z_3 + Z_1Z_3 < 0$. The numbers $\{Z_i \in \mathbb{R}\colon\ i=1,2,3\}$ are referred to as charges; these are also imposed under certain restrictions following from (d). It follows from (1) that the ground state $\inf\mathrm{spec}_d(H_0^0)$ is found by setting $n_1 = n_2 = 1$. For example, if $Z_1 = Z_3 = -1$, $Z_2 = +1$ (eg $Z_1 = Z_2 = -1$, $Z_3 = +1$ is improper by (d)) and $m_1 \leq m_3$, then $\inf\mathrm{spec}_d(H_0^0) = -(4\beta)^{-1}$ in agreement with [3].

[1] R. Juršėnas, to appear in Few-Body Syst. (2013), DOI 10.1007/s00601-012-0525-9

[2] E. A. G. Armour, J. M. Richard, K. Varga, Phys. Rep. **413**, 1 (2005)

[3] A. Martin, J. M. Richard, T. T. Wu, Phys. Rev. A **46**, no. 7, 3697 (1992)

[4] B. Simon, Helv. Phys. Acta **43**, no. 6, 607 (1970)

E-mail: Rytis.Jursenas@tfai.vu.lt